In gravitational and electric fields, the work done to move a mass or charge from one point to another depends on the intial and final position only, not on the path which you take between these positions. The principle of the conservation of energy holds in these fields. Therefore, vector fields which have this property are called **conservative vector fields**.

DEFINITIONS Let $\mathbf{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path C from A to B in D is the same over all paths from A to B. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path independent in D** and the field $\mathbf{F}$ is **conservative on D**.

Please note that A and B are point in three space such that $r(a) = A$ and $r(b) = B$.

Given that the integral of a conservative vector field, ***F***, is no longer dependent on the choice of path from A to B with curve, C, we can write the integral

$$\oint_C F \cdot dr = \int_a^b F \cdot dr$$

But when is a vector field conservative on a domain? Are we going to check every path from A to B and see if the integral over each path is the same? No, we don't have enough time to do such a thing, and frankly, only engineers would want to …. Just kidding! Actually, there is a way, and it will take you back to your childhood days in AP Calculus AB. Do you remember ….

THE FUNDAMENTAL THEOREM OF CALCULUS?

Before we try to find the multivariable analogue to FTC, let us first recall the statement of FTC in single variable calculus.

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2 If f is continuous at every point in $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

So that's what we need! An antiderivative! But wait! There is no derivative of a multivariable function. However, there is … a gradient.

DEFINITION If $\mathbf{F}$ is a vector field defined on D and $\mathbf{F} = \nabla f$ for some scalar function f on D, then f is called a **potential function for $\mathbf{F}$**.

On the next page, we will see why having a potential function is important.

So let us suppose our vector fields was the gradient of some scalar function f on the domain *D*. Then *f* is the multivariable analogue to an antiderivative of **F** and we call it the potential function. Then

$$\int_A^B \nabla f \cdot dr = \int_a^b \left(\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt$$
$$= \int_a^b \frac{df}{dx}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt} + \frac{df}{dz}\frac{dz}{dt}\, dt$$

The integrand should look familiar to you. If you remember the chain rule for the composition of multivariable functions with a path in three space. The theorem below is from Chapter 14, section 4.

> **THEOREM 6—Chain Rule for Functions of Three Independent Variables** If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and
>
> $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

So we can write

$$\int_A^B \nabla f \cdot dr = \int_a^b \frac{df}{dt}\, dt$$

and now we have reduced our problem to the integral over an interval. Therefore, the Fundamental theorem of Calculus holds, and

$$\int_A^B \nabla f \cdot dr = f(x(t), y(t), z(t))\Big|_{t=a}^{t=b}$$
$$= f(r(b)) - f(r(a))$$
$$= f(A) - f(B)$$

□

> **THEOREM 1—Fundamental Theorem of Line Integrals** Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\mathbf{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on a domain D containing C. Then
>
> $$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

If we can find the potential function for a vector field, then integrating becomes trivial.

Example: Suppose $F = \nabla f$ is the gradient function of $f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$. Find the work done by **F** in moving an object along a smooth curve from (1,0,0) to (0,0,2) not passing through the origin.

Solution:

Types of Domains

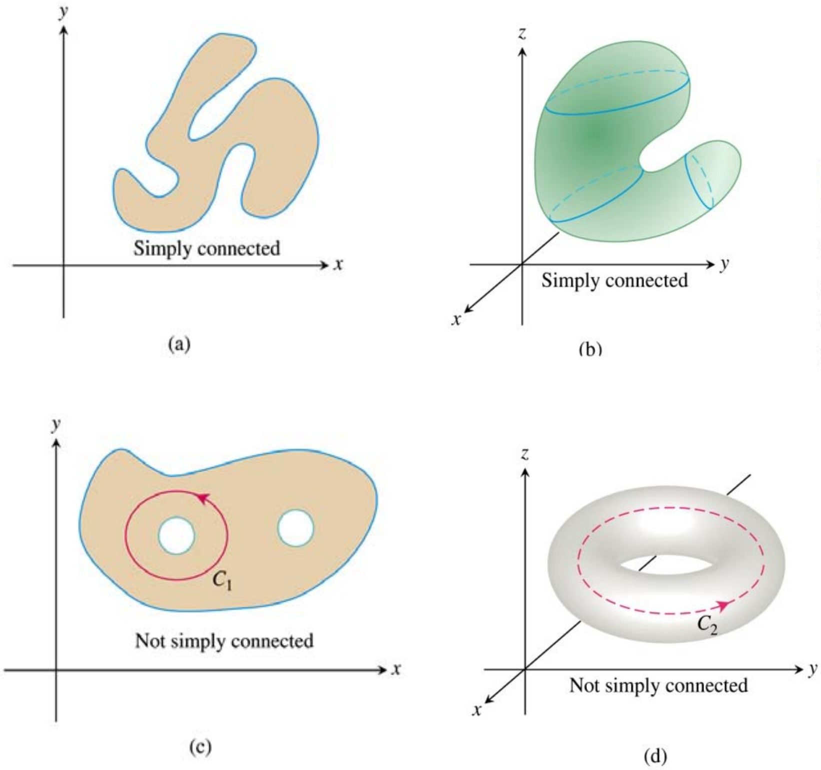


FIGURE 16.22 Four connected regions. In (a) and (b), the regions are simply connected. In (c) and (d), the regions are not simply connected because the curves C_1 and C_2 cannot be contracted to a point inside the regions containing them.

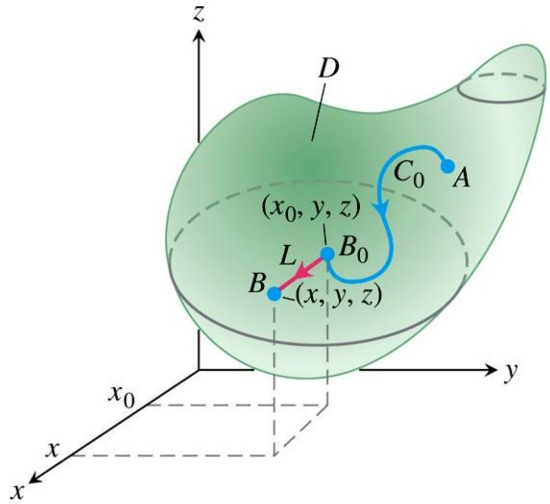
Definitions:

- A domain is said to be ***simply connected*** if every closed loop can be "continuously deformed" to a point.
- In euclidean space, a domain is ***connected*** if and only if it is path connected.
- A domain, *D*, is said to be ***path connected*** if for every two points *A* and *B* in *D*, there is a path from *A* to *B* completely contained in *D.*

  We will soon find out that there is an even easier way to check if a vector field is conservative on a simply connected domain that generalizes to the calculus on complex valued functions. We will also find out that the Fundamental Theorem for Line Integrals becomes an if and only if statement on a connected domain.

Theorem: Let $F=(f_1, f_2, f_3)$ be a continuous vector field on an open connected domain *D*. Then *F* is conservative if and only if *F* has a potential function *f*.

Proof:



Let point $B_0=(x_0, y, z)$ be in the $Ball_\epsilon(B)$. Note only the *x* - coordinate has changed.
Let $L:[x_o, x]\to Ball_\epsilon(B)$ be the line segment from B_0 to B defined by $L(t)=(t, y, z)$ which Since *D* is connected in Euclidean space, then it is path-connected, and therefore, there is a path, $p: I\to D$, from A to B_0 such that its curve $C_0\subset D$. Since, $p(0)=A$ and $p(1)=B_0$ and $L(x_0)=B_0$ and $L(x)=B$, then the concatenation of *p* with *L* yields a path from *A* to *B*.
(see figure to the left)

We have one direction already proven in the Fundamental Theorem of Line Integrals, so it suffices to show the other direction only; namely that if a field is conservative on a connected domain *D*, then it is the gradient of some scalar function.

Let $A\in D$ be a fixed point.
For each $B\in D$, let $r(t): I\to D$ be a path such that $r(0)=A$ and $r(1)=B$.
Suppose *C* is the curve given by the path *r* (*t*).
Define a function, *f*, by

$$f(B)=\int_C F\cdot dr$$

and define *f* (*A*) = 0.Since *F* is conservative over *D*, then the function is well-defined and independent of the choice of *r*.
Now, in order to show that *f* is a potential function for *F*, we need to show that

$$\nabla f = F$$

Suppose $B=(x, y, z)$. Since D is open, there exists an $\epsilon>0$, such that $Ball_\epsilon(B)$ is completely contained in *D*.

Now we because the vector field is conservative,

$$f(B)=\int_C F\cdot dr$$
$$=\int_{C_0} F\cdot dr+\int_L F\cdot dr$$

By construction of *f*,

$$\int_{C_0} F\cdot dr = f(B_0)$$

which is a scalar. Therefore,

$$\frac{\partial}{\partial x} f = \frac{\partial}{\partial x}\left(\int_{C_0} F\cdot dr+\int_L F\cdot dL\right)$$
$$=\frac{\partial}{\partial x}\left(\int_L F\cdot dL\right)$$
$$=\frac{\partial}{\partial x}\left(\int_0^1 (f_1, f_2, f_3)\cdot\frac{dL}{dt}dt\right)$$
$$=\frac{\partial}{\partial x}\left(\int_{x_0}^x (f_1, f_2, f_3)\cdot(1,0,0)dt\right)$$
$$=\frac{\partial}{\partial x}\left(\int_{x_0}^x f_1(t, y, z)dt\right)$$
$$= f_1(x, y, z)$$

A similar argument shows that the other partials of *f* are the coordinate functions for *F*. Therefore, *f* is a potential function for the conservative vector field, *F*. □

Corollary:

THEOREM 3—Loop Property of Conservative Fields The following statements are equivalent.

1. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ around every loop (that is, closed curve C) in D.
2. The field $\mathbf{F}$ is conservative on D.

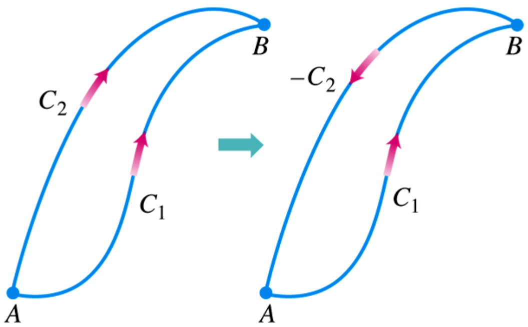


FIGURE 16.24 If we have two paths from A to B, one of them can be reversed to make a loop.

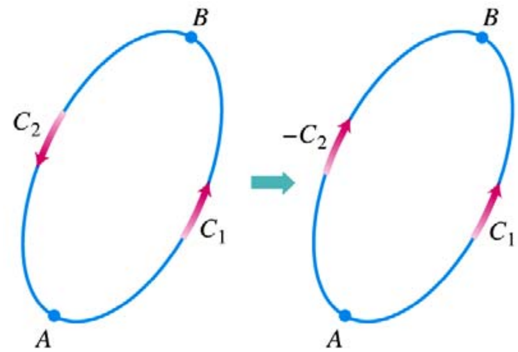


FIGURE 16.25 If A and B lie on a loop, we can reverse part of the loop to make two paths from A to B.

If our conservative vector fields is defined on a simply connected domain, we get the following theorem.

Component Test for Conservative Fields
Let $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be a field on a connected and simply connected domain whose component functions have continuous first partial derivatives. Then, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{2}$$

The proof of this theorem **requires** that the domain be simply connected and is a consequence of Stoke's Theorem, which we will prove later.

<u>Example:</u> Show that $F(x, y, z) = (e^x \cos(y) + yz, xz - e^x \sin(y), xy + z)$ is a conservative vector field and find a potential function for it.

Step 1: Is its natural domain simply connected?
Step 2: Solve the equations given above.
Step 3: Find a potential function for it.

<u>Differential Forms</u>

In this class we will only study 1-forms. However, the concept of differential form are introduced because eventually, it will be necessary to perform calculus on n-dimensional locally Euclidean spaces we call ***differentiable manifolds.*** The differential form will eventually allow us to discard the coordinate system and give us a unified approach to integrands over manifolds.

DEFINITIONS Any expression $M(x, y, z)\,dx + N(x, y, z)\,dy + P(x, y, z)\,dz$ is a **differential form**. A differential form is **exact** on a domain D in space if

$$M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

for some scalar function f throughout D.

Component Test for Exactness of $M\,dx + N\,dy + P\,dz$

The differential form $M\,dx + N\,dy + P\,dz$ is exact on a connected and simply connected domain if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

This is equivalent to saying that the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative.

<u>Example:</u> Show that the differential form

$$2xdx + 2ydy + 2zdz$$

is exact. Then compute $\int_{(0,0,0)}^{(2,3,-6)} 2xdx + 2ydy + 2zdz$.