Parametrizing Surfaces

Just as we parametrized curves in space with vector valued functions in one time variable,

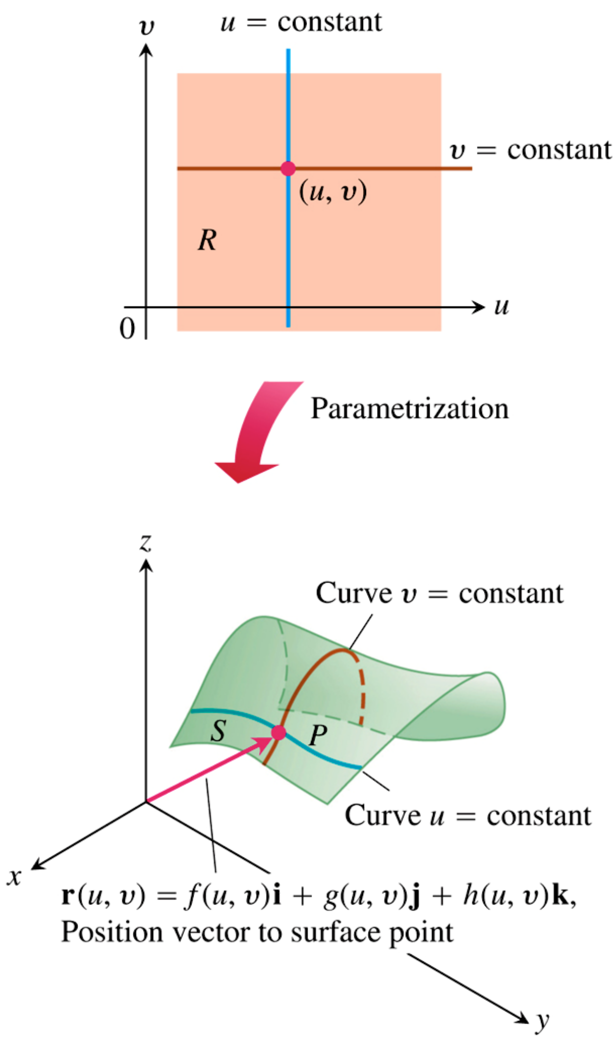
$$r(t) = (x(t), y(t), z(t))$$

we can parametrize surfaces embedded in 3-space with vector valued functions in two time variables

$$r(u,v) = (x(u,v), y(u,v), z(u,v)).$$

Note that using only one variable for $r(t)$ and assuming the coordinate function were differentiable with nonvanishing derivative (smooth), ensured that locally, the graph of r was in one to one correspondence to a small open interval.

A similar argument, given that the coordinate functions are one to one on the interior of the domain of r, ensures that the graph of r will be in one to one correspondence with an open connected domain in $\mathbb{R}^2$.



Note that using only one variable for $r(t)$ and assuming the coordinate function were differentiable with nonvanishing derivative (smooth), ensured that locally, the graph of r was in one to one correspondence to a small open interval.

A similar argument, given that the coordinate functions are one to one on the interior of the domain of r, ensures that the graph of r will be in one to one correspondence with an open connected domain in $\mathbb{R}^2$.

Example 1: Suppose we were asked to find a parametrization for the cone $z = \sqrt{x^2 + y^2}$.

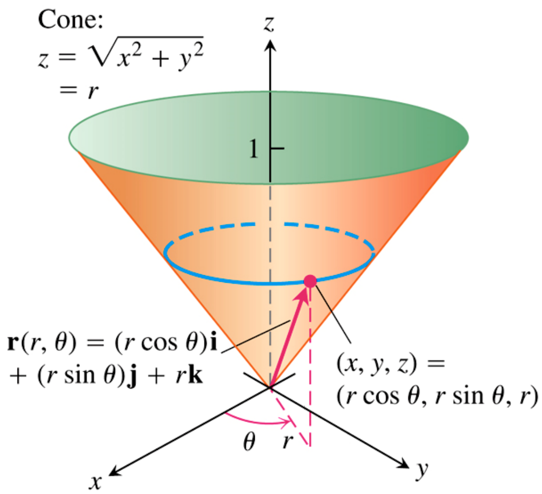


FIGURE 16.38 The cone in Example 1 can be parametrized using cylindrical coordinates.

$$r(r,\theta) = (r\cos(\theta), r\sin(\theta), r) \quad 0 \le r \le 1, 0 \le \theta \le 2\pi$$

Example 2: We can just as easily find a parametrization for a sphere.

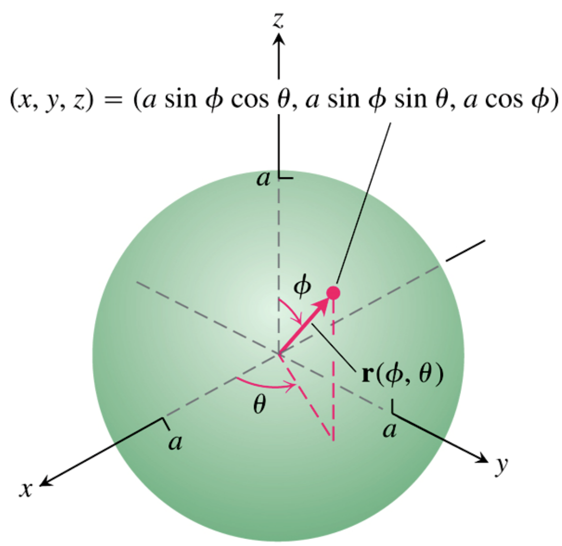


FIGURE 16.39 The sphere in Example 2 can be parametrized using spherical coordinates.

Example 3: Find a parametrization for the surface

$$x^2 + (y-3)^2 = 9\ , \qquad 0 \le z \le 5$$

Substituting $x = r\cos(\theta)$ and $y = r\cos(\theta)$, yields

$$\begin{aligned} x^2 + y^2 - 6y + 9 &= 9 \\ x^2 + y^2 - 6y &= 0 \\ r^2\cos^2(\theta) + r^2\sin^2(\theta) - 6r\sin(\theta) &= 0 \\ r^2 - 6r\sin(\theta) &= 0 \\ r(r - 6\sin(\theta)) &= 0 \\ r &= 0 \text{ or } r = 6\sin(\theta) \end{aligned}$$

Hence, by subsituting this equation into $x = r\cos(\theta)$ we get

$$x = 6\sin(\theta)\cos(\theta) = 3\sin(2\theta)$$

Similarly, we can get the other two coordinate equations

$$\begin{aligned} y &= 6\sin(\theta)\sin(\theta) = 6\sin^2(\theta) \\ z &= z \end{aligned}$$

The domain of the function is $(\theta, z) = [0,\pi]\times[0,5]$. Why can't $0 \le \theta \le 2\pi$?

The answer is because the function needs to be one to one. The *x* and *y* coordinate functions have a period of π. Therefore, allowing the domain to go all the way to 2π would violate this property.

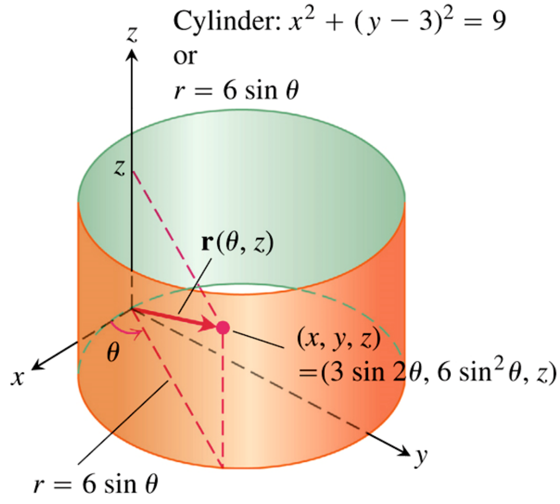


FIGURE 16.40 The cylinder in Example 3 can be parametrized using cylindrical coordinates.

Smoothness

We have used the term smooth many times in this class. So far it has meant that the component functions or coordinate functions are differentiable and have nonvanishing derivative. This becomes more difficult to express with multivariable, vector-valued functions (vector fields), like the ones above.
Let us define

$$r(u,v) = (f(u,v), g(u,v), h(u,v))$$

where f is the *x* coordinate function, *g* is the *y* coordinate function, and *h* is the *z* coordinate function. Then

$$r_u = \left(\frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}, \frac{\partial h}{\partial u}\right) \text{ and } r_v = \left(\frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v}\right)$$

> **DEFINITION** A parametrized surface $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is **smooth** if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v$ is never zero on the interior of the parameter domain.

We will investigate this property further. Given two vectors *a* and *b*, what do we know about their cross product? Well, it's perpendicular to both *a* and *b*. What else? Hold on! The magnitude of $a \times b$, $|a \times b|$, is the area of the parallelogram spanned by *a* and *b*. So requiring $a \times b$ to never be zero is requiring *a* and *b* to never be collinear. In the case of r_u and r_v, we are asking the the tangent vector of *r* in the *u* direction never be on the same line as the tangent vector of *r* in the *v* direction.

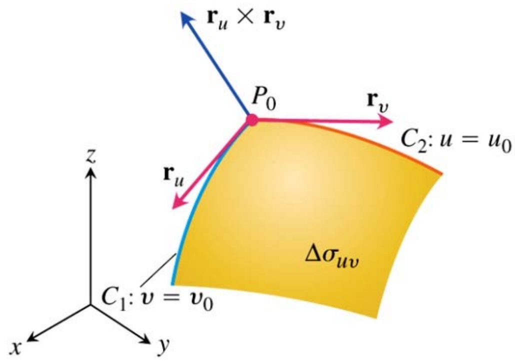


FIGURE 16.42 A magnified view of a surface patch element $\Delta\sigma_{uv}$.

This would violate the one to one property. Thus, this property ensures our surface locally looks like a piece piece of the plane.

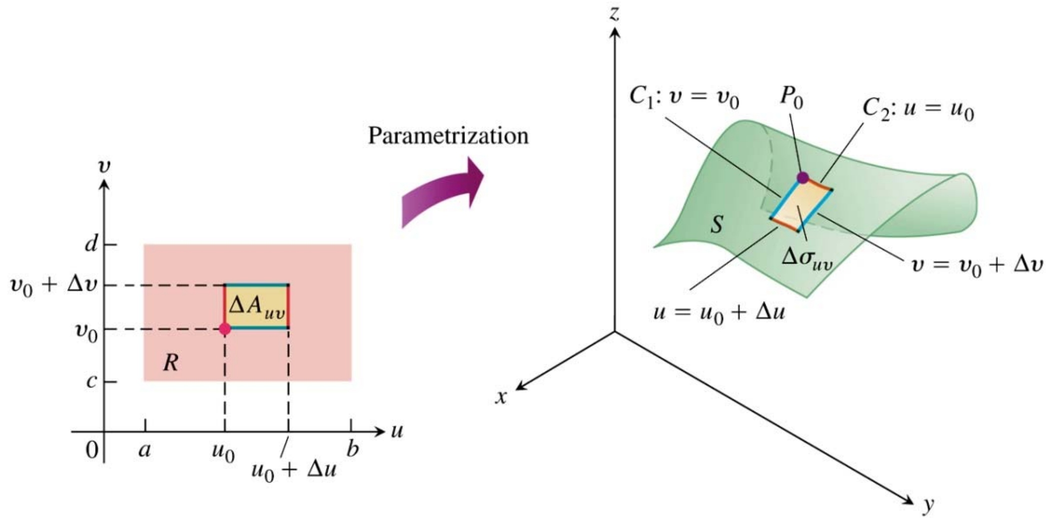


FIGURE 16.41 A rectangular area element ΔA_{uv} in the uv-plane maps onto a curved patch element $\Delta\sigma_{uv}$ on S.

Surface Area

Due to the smoothness of the parametrization, for every point on the surface, p , there is a neighborhood of p which can be "flattened" out to look like an open rectangle in the plane. This means the "curviness" is not required for this small patch. Therefore, the area of the patch is given by the area of the rectangle.

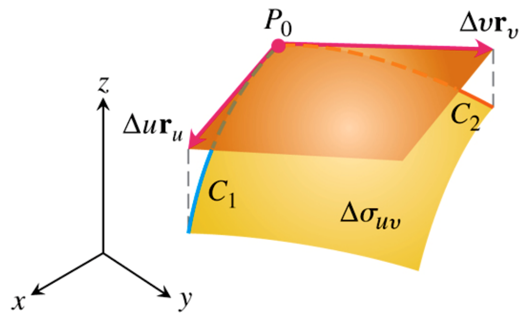


FIGURE 16.43 The area of the parallelogram determined by the vectors $\Delta u \mathbf{r}_u$ and $\Delta v \mathbf{r}_v$ is defined to be the area of the surface patch element $\Delta\sigma_{uv}$.

This small patch around my point is the rectangle given by the vectors $\Delta u r_u$ and $\Delta v r_v$. The area is, therefore, given by $|\Delta u r_u \times \Delta v r_v|$. By linearity of cross product

$$|\Delta u r_u \times \Delta v r_v| = |r_u \times r_v| \Delta u \Delta v$$

Thus, the surface area of the surface can be achieved by

DEFINITION The **area** of the smooth surface

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad a \le u \le b, \quad c \le v \le d$$

is

$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| \, dA = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv. \qquad (4)$$

Exercises:

1) Find the surface area of the cone $r(r,\theta) = (r\cos(\theta), r\sin(\theta), r) \quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi$.

2) Find the surface area of a sphere of radius *r*.