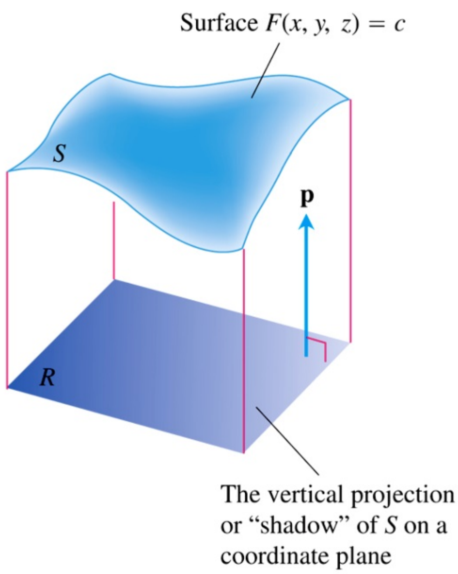
Implicit Surfaces

Most surfaces are written as the preimage of a number under a continuous function. In other words, they are the level surfaces of a function. We have studied these before when we studied quadric surfaces. However, to find the surface area of some implicit surfaces may be difficult if we don't have a parametrization.

First we need to find a property that ensures one-to-oneness, like we did for parametrized surfaces. To do this we will look at the projection of the implicit surface on a plane



If we can find a parametrization for the surface, *S*, then we can compute the surface area by evaluating a double integral over the region *R* on a coordinate plane.

The vector ***p*** that you see is the normal vector to the plane. The surface given by the level surface of the function $F(x,y,z)=c$, is considered smooth if *F* is differentiable and ∇F is nonzero and continuous on *S*. Recall that the ∇F is always normal to the level surface at a point. If we further assume that $\nabla F \cdot p \neq 0$, then we would be ensuring that the normal to the level surface is not inline with the plane onto which it's projecting. This would mean that the surface is folding back on itself.

Assume that the plane is the *xy*-coordinate plane, then the normal unit vector would be (0,0,1). Our properties ensure that

$$\nabla F \cdot (0,0,1) \neq 0$$

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) \cdot (0,0,1) \neq 0$$

$$\frac{\partial F}{\partial z} \neq 0$$

If you study the theory of Calculus, you may run across the Implicit Function Theorem, which in a nutshell states that if the Jacobian of a multivariable, vector-valued function over the real numbers has a non-zero determinant at a point, then the locus of the function is locally the graph of a function.

In this case, we only need the partial with respect to *z* to be nonzero, and the Implicit Function Theorem says that we can write $z=h(x,y)$, where *u* and *v* live on *R*. We may not have *h* explicity, but we know it exists, and this allows us to parametrize our surface. $r(x,y)=(x,y,h(x,y))$. Now we can revert to the previous case of finding the surface area.

We need to calculate $r_x = (1, 0, \frac{\partial h}{\partial x})$ and $r_y = (0, 1, \frac{\partial h}{\partial y})$. However, the chain rule from 14.4 tells us that

$$\frac{\partial h}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \text{ and } \frac{\partial h}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Therefore, we can substitute these into $r_x = (1, 0, \frac{\partial h}{\partial x})$ and $r_y = (0, 1, \frac{\partial h}{\partial y})$ and take the cross product to get

$$\begin{aligned} r_u \times r_v &= \left(\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}, -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}, 1 \right) \\ &= \frac{1}{\frac{\partial F}{\partial z}} \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) \\ &= \frac{1}{\frac{\partial F}{\partial z}} \nabla F \\ &= \frac{\nabla F}{\nabla F \cdot (0, 0, 1)} \end{aligned}$$

However, choosing $(0,0,1) = \boldsymbol{p}$ was completely arbitrary. We could have just as easily concluded this if $\boldsymbol{p}$ was (0,1,0) and $\frac{\partial F}{\partial y} \neq 0$. Thus, we have the following formula for the surface area of an implicit surface.

Formula for the Surface Area of an Implicit Surface

The area of the surface $F(x, y, z) = c$ over a closed and bounded plane region R is

$$\text{Surface area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} \, dA, \qquad (7)$$

where $\mathbf{p} = \mathbf{i}, \mathbf{j}$, or $\mathbf{k}$ is normal to R and $\nabla F \cdot \mathbf{p} \neq 0$.

Below is the original statement of the Implicit Function Theorem

Given

$F_1(x, y, z, u, v, w) = 0$ (1)
$F_2(x, y, z, u, v, w) = 0$ (2)
$F_3(x, y, z, u, v, w) = 0,$ (3)
if the determinantof the Jacobian

$$|J\, F(u, v, w)| = \left|\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)}\right| \neq 0, \quad (4)$$

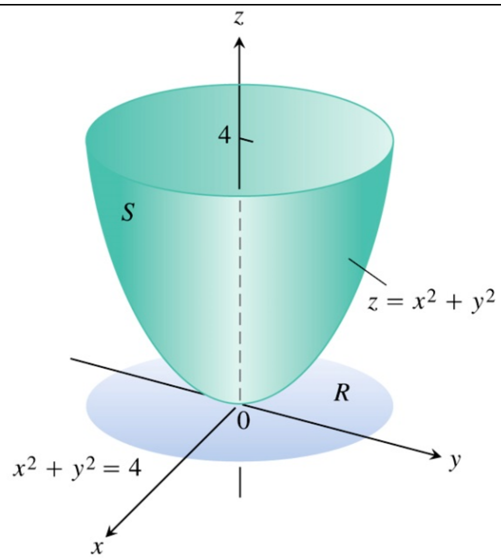
then u, v, and w can be solved for in terms of x, y, and z and partial derivatives of u, v, w with respect to x, y, and z can be found by differentiating implicitly.

More generally, let A be an open set in $\mathbb{R}^{n+k}$ and let $f : A \to \mathbb{R}^n$ be a C^r function. Write f in the form $f(x, y)$, where x and y are elements of $\mathbb{R}^k$ and $\mathbb{R}^n$. Suppose that (a, b) is a point in A such that $f(a, b) = 0$ and the determinant of the $n \times n$ matrix whose elements are the derivatives of the n componentfunctions of f with respect to the n variables, written as y, evaluated at (a, b), is not equal to zero. The latter may be rewritten as

$$\operatorname{rank}(D\, f(a, b)) = n. \quad (5)$$

Then there exists a neighborhood B of a in $\mathbb{R}^k$ and a unique C^r function $g : B \to \mathbb{R}^n$ such that $g(a) = b$ and $f(x, g(x)) = 0$ for all $x \in B$

from: Wolfram Alpha



Example 1:

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.

Solution 1: This is an implicitly defined surface, so we need to compute the gradient.

$$\nabla F = (2x, 2y, -1)$$

We see that $\frac{\partial F}{\partial z} \neq 0$, so we will let $\boldsymbol{p} = (0,0,1)$.

$$\iint_R \frac{|\nabla F|}{|\nabla F \cdot p|} dA = \iint_{x^2+y^2 \le 4} \sqrt{4x^2 + 4y^2 + 1}\, dx dy$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\, r dr d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}(4r^2 + 1)^{\frac{3}{2}} \Big|_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}(17^{\frac{1}{2}} - 1) d\theta = \frac{\pi}{6}(17\sqrt{17} - 1)$$