Review of Stoke's Theorem

Many of you have asked about the difference between Stokes' Theorem and Green's Theorem, so I will try to elucidate for you. First, let us look at the statements.

THEOREM 5—Green's Theorem (Circulation-Curl or Tangential Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\mathbf{F}$ around C equals the double integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ over R.

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy \tag{4}$$

Counterclockwise circulation   Curl integral

If you notice, the curve *C* is a planar curve living in the *xy*-plane. However, Green's Theorem relates the circulation of ***F*** around *C* is the double integral of the *k*-th component of the curl of ***F*** over the region bounded by *C*.

THEOREM 6—Stokes' Theorem Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then the circulation of $\mathbf{F}$ around C in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of $\nabla \times \mathbf{F} \cdot \mathbf{n}$ over S.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma \tag{4}$$

Counterclockwise circulation   Curl integral

In Stokes' Theorem, the curve, *C*, is a space curve and the region it bounds is an orientable smooth surface. As in Green's Theorem, Stokes' Theorem relates the circulation around the curve with the double integral of the curl of the vector field over the region bounded by the curve.

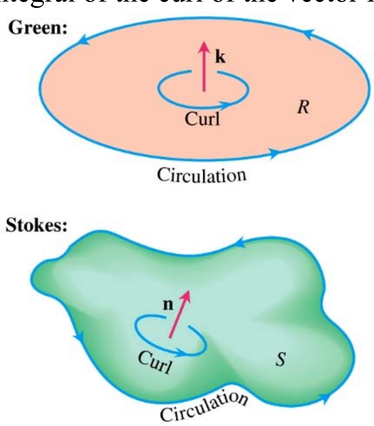

FIGURE 16.57 Comparison of Green's Theorem and Stokes' Theorem.

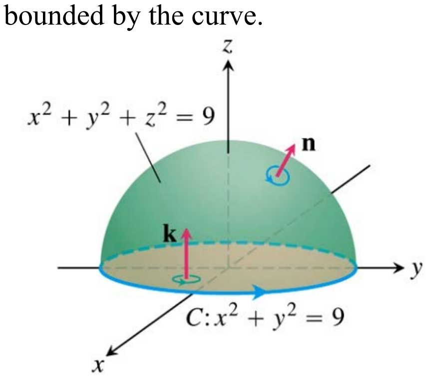

FIGURE 16.58 A hemisphere and a disk, each with boundary C (Examples 2 and 3).

THEOREM 7—Curl F = 0 Related to the Closed-Loop Property If $\nabla \times \mathbf{F} = \mathbf{0}$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

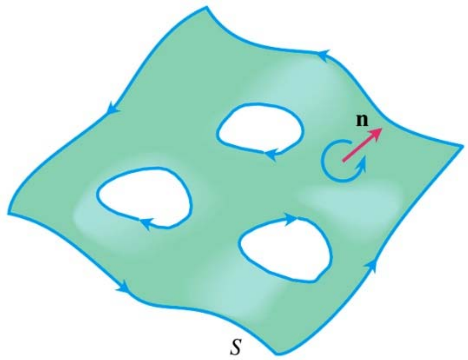

FIGURE 16.65 Stokes' Theorem also holds for oriented surfaces with holes.

The following diagram summarizes the results for conservative fields defined on connected, simply connected open regions.

$\mathbf{F}$ conservative on D $\Longleftrightarrow$ (Theorem 2, Section 16.3) $\mathbf{F} = \nabla f$ on D

$\mathbf{F}$ conservative on D $\Updownarrow$ (Theorem 3, Section 16.3) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ over any closed path in D

$\mathbf{F} = \nabla f$ on D $\Downarrow$ (Vector identity (Eq. 8) (continuous second partial derivatives)) $\nabla \times F = \mathbf{0}$ throughout D

$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ over any closed path in D $\Longleftarrow$ (Theorem 7 Domain's simple connectivity and Stokes' Theorem) $\nabla \times F = \mathbf{0}$ throughout D

<u>Divergence Theorem</u>

Remember in section 16.4, Green's Theorem, we learned about the divergence, or flux density of a vector field, and one of the versions of Green's Theorem, related outward flux of a vector field across a curve with the divergence of the vector field over the region bounded by the curve. However, the curve and region were planar.

THEOREM 4—Green's Theorem (Flux-Divergence or Normal Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the outward flux of $\mathbf{F}$ across C equals the double integral of div $\mathbf{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}_{\text{Outward flux}} = \oint_C M\,dy - N\,dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy}_{\text{Divergence integral}} \qquad (3)$$

The last theorem of the class relates the outward flux of a vector field, ***F*****,** across a smooth orientable surface with the integral of the divergence of the vector field, $\nabla\cdot F$, over the three-dimensional domain that was bounded by the surface.

THEOREM 8—Divergence Theorem Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives, and let S be a piecewise smooth oriented closed surface. The flux of $\mathbf{F}$ across S in the direction of the surface's outward unit normal field $\mathbf{n}$ equals the integral of $\nabla\cdot\mathbf{F}$ over the region D enclosed by the surface:

$$\underbrace{\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma}_{\text{Outward flux}} = \underbrace{\iiint_D \nabla\cdot\mathbf{F}\,dV.}_{\text{Divergence integral}} \qquad (2)$$

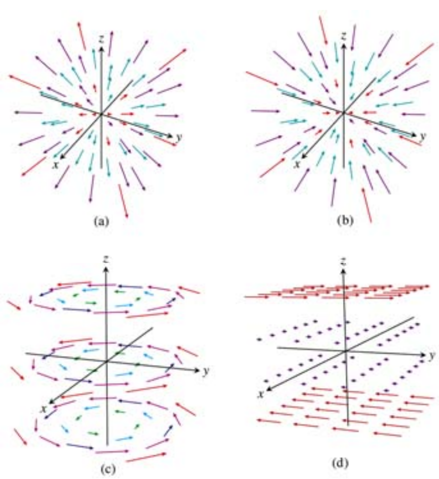

FIGURE 16.67 Velocity fields of a gas flowing in space (Example 1).

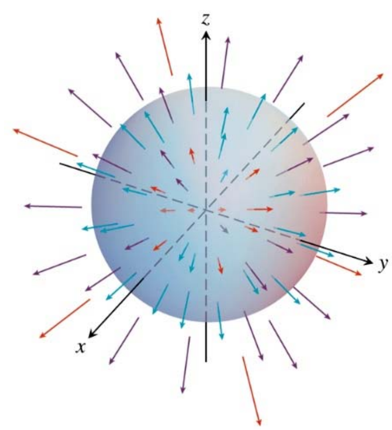

FIGURE 16.68 A uniformly expanding vector field and a sphere (Example 2).

We will sketch the proof for a particular case of a surface whose projection onto the *xy*-plane is a region, R_{xy}, for which the *z*-coordinate of the surface can be written as a function of (*x*, *y*) in R_{xy}. For general regions, we will break up the smooth orientable surface into piecewise surfaces that do satisfy this property.

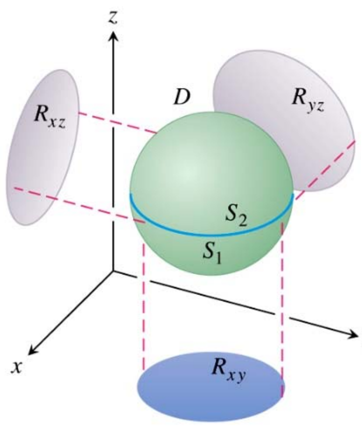

FIGURE 16.69 We prove the Divergence Theorem for the kind of three-dimensional region shown here.

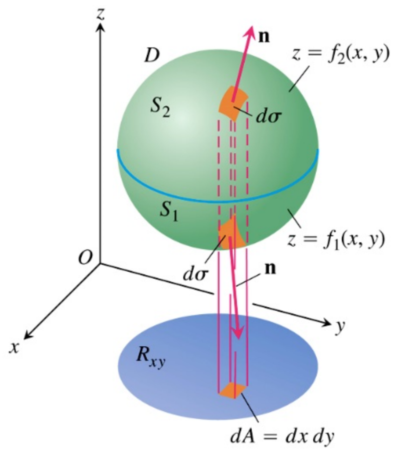

FIGURE 16.71 The region *D* enclosed by the surfaces S_1 and S_2 projects vertically onto R_{xy} in the *xy*-plane.

If $F(x,y,z)=(M(x,y,z),N(x,y,z),P(x,y,z))$ then we will only show

$$\iint_S P\cdot n \;\; dS = \iiint_D \frac{\partial P}{\partial z}dV$$

The lower surface will be $f_1(x,y)$ and the upper surface will be written as $f_2(x,y)$. The rest of the components of the integral can be proven the same way.

$$\begin{aligned}\iiint_D \frac{\partial P}{\partial z}dV &= \iint_R \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial P}{\partial z}dzdxdy\\ &= \iint_R P(x,y,f_2(x,y)) - P(x,y,f_1(x,y))dxdy\end{aligned}$$

Since the surface can be parameterized by the *xy* plane, then the surface area element will be (0,0,1) for the upper part of the surface and (0,0,-1) for the lower part of the surface. Therefore, the flux across the surface can be computed by

$$\iint_{S_2} PdS = \iint_R P(x,y,f_2(x,y))dxdy$$

and

$$\iint_{S_1} PdS = -\iint_R P(x,y,f_1(x,y))dxdy$$

Adding up the two flux, we get the flux across *S*. Therefore, $\iiint_D \frac{\partial P}{\partial z}dV = \iint_S P\cdot n \;\; dS$. Using this same idea, we prove the other components and then the general form of the Divergence Theorem.

Green's Theorem and Its Generalization to Three Dimensions

Normal form of Green's Theorem:	$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$
Divergence Theorem:	$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$
Tangential form of Green's Theorem:	$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA$
Stokes' Theorem:	$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$

A Unifying Fundamental Theorem

The integral of a differential operator acting on a field over a region equals the sum of the field components appropriate to the operator over the boundary of the region.

THE END!!!!