## **<u>Arithmetic Series</u>**

There are some types of sequences and series that arise naturally, and therefore, have special names.

For example, your Instagram starts off with one follower, then each week account gains 2 more than the number of followers gained the previous week, the number of new followers per week would be the sequence:

| Week 1 | Week 2 | Week 3 | Week 4 | Week 5 | Week 6 | Week 7 | Week 8 | Week 9 | Week 10 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 |

The sequence would be

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots$$

We call this sequence *arithmetic* because the difference between any two consecutive terms is constant. In other words, to get from one term to the next, you need to add 2.

$$a_1 = 1$$
$$a_2 = 1 + 2 = 3$$
$$a_3 = 3 + 2 = 5$$

When finding the formula for the nth term, arithmetic sequences are lines, $f(x) = mx + b$, but with an *n* instead of an x. We could write a function for the number of Instagram followers gained on the nth week:

$$a_n = 2n - 1$$

However, if we wanted to know how many total Instagram follower you had on the 10$^{th}$ week, we would have to add the terms. One could add them in order

$$1+3+5+7+9+11+13+15+17+19$$

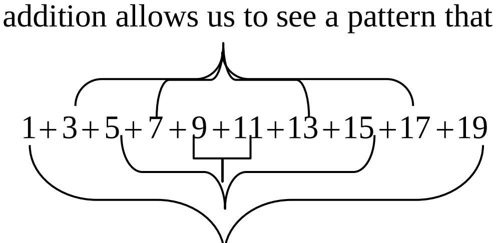
term by term, but even writing it all out is time consuming.

We already learned about *sigma notation*, which makes the expression shorter:

$$\sum_{n=1}^{10} 2n - 1 = 1+3+5+7+9+11+13+15+17+19$$

Sometimes there are tricks to finding the sum of terms of a sequence that make the computation more efficient.

The Gaussian method of addition allows us to see a pattern that is unique to arithmetic sequences.



Add the pairs indicated:

There are 5 pairs of 20 which yields 100. You have 100 Instagram followers on Week 10. This pattern emerges in every arithmetic sequence because the second term is 2 more than the first term and the second to last term is 2 less than the last term. We added 2 in one direction and subtracted it in the other direction.

**<u>Question:</u>** How many Instagram followers would you have in one year?

(Work it out: I'll tell you later)

Example:

Let $b_n = 3n - 4$.

Find $\sum_{n=1}^{100} b_n = \sum_{n=1}^{100} 3n - 4$

$$= -1 + 2 + 5 + \cdots + 287 + 290 + 293 + 296$$

$$= (-1 + 296) + (2 + 293) + (5 + 290) + \cdots$$

We just added them in a different order. There are 50 pairs that add up to 295, so the sum is (50)(295) = 14750

So the formula for the sum of the n terms of an arithmetic sequence is given by

(first term + last term) (half of number of terms)

**<u>Check for Understanding 1:</u>**

1. $\sum_{n=1}^{53} 5n + 1$

2. $\sum_{n=1}^{23} n - 4$

(Answers are on the last page)

## Geometric Series

**Warm up :** Distribute the products below

1. $(1-x)(1+x+x^2)=$
2. $(1-x)(1+x+x^2+x^3)=$
3. $(1-x)(1+x+x^2+x^3+\cdots+x^{n-1}+x^n)=$

Using sigma notation, we can write

$$\sum_{k=0}^{n} x^k = 1+x+x^2+x^3+x\cdots+x^{n-1}+x^n$$

Therefore , if we divide both sides of the equation in Warm-up #3 by (1-*x*) on both sides, we get:

$$\sum_{k=0}^{n} x^k = 1+x+x^2+x^3+x\cdots+x^{n-1}+x^n = \frac{1-x^{n+1}}{1-x}$$

and, thus,

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty}\sum_{k=0}^{n} x^k = \lim_{n\to\infty}\frac{1-x^{n+1}}{1-x}$$

If $|x| < 1$, what does $x^{n+1} \to$ ______, as $n \to \infty$?

If you're stuck, imagine that $x=\frac{1}{2}$, then compute $(\frac{1}{2})^{n+1} \to$ ______, as $n \to \infty$?

**Geometric Series Test:** If $|x| < 1$.

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

Example:

Compute $\sum_{n=0}^{\infty}\frac{1}{2^n}$

Does it matter that it's an *n* and not a *k*? No.

So, first we need to see that it's a geometric series, by making it of the form "sum of powers of __"

$\sum_{n=0}^{\infty}\frac{1}{2^n} = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ Since $x=\frac{1}{2}<1$, we can use the geometric series test. So

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

**Check for Understanding 2:**

Using what we showed above, find the following

a) $\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n$

b) $\sum_{n=0}^{\infty}\frac{1}{5^n}$

**Answers:**

**Question:** 2704

**Check for Understanding 1:**

1. $\sum_{n=1}^{53} 5n+1=(6+266)(\frac{53}{2})=7208$ 2. $\sum_{n=1}^{23} n-4=(-3+19)(\frac{23}{2})=184$

**Warm up :** Distribute the products below

1. $(1-x)(1+x+x^2)=1+x-x+x^2-x^2-x^3=1-x^3$
2. $(1-x)(1+x+x^2+x^3)=1-x^4$
3. $(1-x)(1+x+x^2+x^3+\cdots+x^{n-1}+x^n)=1-x^{n+1}$

**Fill in Blanks:**

If $|x| < 1$, what does $x^{n+1}\to 0,$ as $n\to\infty$?

**Check for Understanding 2:**

Using what we showed above, find the following

a) $\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n=\frac{1}{1-\frac{3}{4}}=\frac{1}{\frac{1}{4}}=4$ b) $\sum_{n=0}^{\infty}\frac{1}{5^n}=\frac{1}{1-\frac{1}{5}}=\frac{1}{\frac{4}{5}}=\frac{5}{4}$