also be sure to carry according to the base in which we multiply.

$$\begin{array}{r} {}^{1\,2}_{1\,2} \\ 241_6 \\ \times \quad 34_6 \\ \hline 1444_6 \\ +\ 12030_6 \\ \hline 13514_6 \end{array}$$

Here are a few more examples of base number multiplication. Walk through these examples on your own to be sure you understand the process of base number multiplication.

$$\begin{array}{r} 23_4 \\ \times\ 11_4 \\ \hline 313_4 \end{array} \qquad \begin{array}{r} 67_9 \\ \times\ 18_9 \\ \hline 1372_9 \end{array} \qquad \begin{array}{r} 134_5 \\ \times\ 43_5 \\ \hline 13022_5 \end{array} \qquad \begin{array}{r} 6A9_{14} \\ \times\ 32B_{14} \\ \hline 178A91_{14} \end{array}$$

Exercises

9.4.1 Find each product within the indicated number base.

(a) $6_8 \cdot 7_8$
(b) $12_3 \cdot 201_3$
(c) $76_8 \cdot 57_8$
(d) $11_5 \cdot 3113_5$
(e) $192_{11} \cdot 3AA_{11}$
(f) $4213_6 \cdot 1215_6$

9.5 Base Number Division and Divisibility

Problems

Problem 9.9:
(a) Find the quotient and remainder when 5_7 is divided into 4 parts.
(b) Find the quotient and remainder when 13_7 is divided into 4 parts.
(c) Find the quotient and remainder when 26_7 is divided into 4 parts.
(d) When 536_7 is divided into 4 equal parts, what is the value of each part?

Problem 9.10: Determine which of the following base numbers are multiples of 6.
(a) 11110_2
(b) 313_4
(c) $1AC_{20}$
(d) 5241_7

Problem 9.9: Find the value of $536_7 \div 4_7$.

Solution for Problem 9.9:

Long division uses the division theorem, addition, subtraction, and multiplication to make the process of division easier. There is no reason that we can't use long division with base numbers. We must just keep in mind the ways in which addition, subtraction, and multiplication differ in base number arithmetic. In the example to the right, we divide 536_7 into 4_7 equal parts to find that each has the value 125_7.

$$\begin{array}{r|l} & 125_7 \\ \hline 4_7 & 536_7 \\ & \underline{4} \\ & 13 \\ & \underline{11} \\ & \ \ 26 \\ & \ \ \underline{26} \\ & \ \ \ \ 0 \end{array}$$

□

Let's take a look at a few more examples of base number division, including remainders. Walk through these examples on your own to be sure you understand the process of base number division.

$$\begin{array}{r|l} & 347_8 \text{ R } 2_8 \\ \hline 3_8 & 1267_8 \end{array} \qquad \begin{array}{r|l} & 301_5 \text{ R } 11_5 \\ \hline 12_5 & 4123_5 \end{array} \qquad \begin{array}{r|l} & 2B6_{13} \text{ R } 46_{13} \\ \hline 1A7_{13} & 52B19_{13} \end{array}$$

Problem 9.10: Determine which of the following base numbers are multiples of 6.

(a) 11110_2 (c) $1AC_{20}$
(b) 313_4 (d) 5241_7

Solution for Problem 9.10: We can take a number of approaches to problems like these.

(a) We use long division to see if 11110_2 leaves a remainder when divided by $6 = 110_2$:

$$\begin{array}{r|l} & \ \ 101_2 \\ \hline 110_2 & 11110_2 \\ & \underline{110} \\ & \ \ 110 \\ & \ \ \underline{110} \\ & \ \ \ \ \ 0 \end{array}$$

So, 11110_2 is a multiple of 6.

(b) All multiples of 6 are even numbers. However, we can tell from the units digit of 313_4 that it is odd since each of the other digits represents bundling of even values:

$$313_4 = 3 \cdot 4^2 + 1 \cdot 4^1 + 3 \cdot 4^0 = 2(6 \cdot 4^1 + 2 \cdot 4^0) + 3.$$

Therefore, 313_4 is odd, so it is not a multiple of 6.

(c) We convert $1AC_{20}$ to decimal form to see if it is a multiple of 6:

$$1AC_{20} = 1 \cdot 20^2 + 10 \cdot 20^1 + 12 \cdot 20^0 = 400 + 200 + 12 = 612.$$

Since $612 \div 6 = 102$ without a remainder, we know $1AC_{20}$ is a multiple of 6.

(d) We subtract multiples of 6 out of each base 7 digit bundle fairly easily since $7 - 1 = 6$. If we get to 0 by subtracting multiples of 6, then 52417_7 is a multiple of 6.

$$\begin{aligned} 52417_7 &= 5 \cdot 7^3 + 2 \cdot 7^2 + 4 \cdot 7^1 + 1 \cdot 7^0 \\ &= 5 \cdot 7 \cdot 7^2 + 2 \cdot 7^2 + 4 \cdot 7^1 + 1 \cdot 7^0 \\ &= 5(6+1)7^2 + 2 \cdot 7^2 + 4 \cdot 7^1 + 1 \cdot 7^0 \\ &= 5 \cdot 6 \cdot 7^2 + 5 \cdot 7^2 + 2 \cdot 7^2 + 4 \cdot 7^1 + 1 \cdot 7^0 \end{aligned}$$

Since $5 \cdot 6 \cdot 7^2$ is a multiple of 6, we subtract it out from the total and continue since the new total will leave the exact same remainder as 5241_7 has when divided by 6:

$$\begin{aligned} 5 \cdot 7^2 + 2 \cdot 7^2 + 4 \cdot 7^1 + 1 \cdot 7^0 &= 7 \cdot 7^2 + 4 \cdot 7^1 + 1 \cdot 7^0 \\ &= 7 \cdot 7 \cdot 7^1 + 4 \cdot 7^1 + 1 \cdot 7^0 \\ &= 7(6+1)7^1 + 4 \cdot 7^1 + 1 \cdot 7^0 \\ &= 7 \cdot 6 \cdot 7^1 + 7 \cdot 7^1 + 4 \cdot 7^1 + 1 \cdot 7^0 \end{aligned}$$

Again, we subtract a multiple of 6 from the total: $7 \cdot 6 \cdot 7^1$. We continue to see if the remaining integer is a multiple of 6:

$$\begin{aligned} 7 \cdot 7^1 + 4 \cdot 7^1 + 1 \cdot 7^0 &= 11 \cdot 7^1 + 1 \cdot 7^0 \\ &= 11 \cdot 7 \cdot 7^0 + 1 \cdot 7^0 \\ &= 11(6+1)7^0 + 1 \cdot 7^0 \\ &= 11 \cdot 6 \cdot 7^0 + 11 \cdot 7^0 + 1 \cdot 7^0 \end{aligned}$$

Now when we subtract out $11 \cdot 6 \cdot 7^0$, the only thing left is $11 \cdot 7^0 + 1 \cdot 7^0 = 11 + 1 = 12$, which is a multiple of 6, so 5241_7 is a multiple of 6.

Note that over the course of subtraction above, we add each leftmost digit to the next leftmost digit. The result is the sum of the digits: $5 + 2 + 4 + 1 = 12$. This means that a base 7 integer is a multiple of 6 if and only if the sum of its digits is a multiple of 6. We explore this kind of "divisibility rule" in more detail later in this book.

We used four different methods to solve the four parts of this problem. Sometimes we cleverly applied relationships between integers. However, standard division always works with these kinds of problems. □

Exercises

9.5.1 Perform the indicated division within the given base. Include any remainders.

(a) $134_9 \div 7_9$
(b) $11111_2 \div 101_2$
(c) $1444_6 \div 31_6$
(d) $4516_8 \div 43_8$
(e) $81818_{11} \div 81_{11}$
(f) $9A71B_{16} \div 3E9_{16}$

9.5.2 Is 2246_8 divisible by 16_8?

9.5.3 Is 4554_7 divisible by 11_7?

9.5.4 Determine which of the following base numbers are multiples of 3.

(a) 1221_3
(b) 334_5
(c) 4113_6
(d) 7881_9

9.6 Summary

Concept: Addition, subtraction, multiplication, and exponentiation are just methods of fast counting. Since base number systems provide us with ways to write counting and arithmetic results, we can use arithmetic in all base number systems.

The secret to base number arithmetic is that there is no secret. We just need to be careful and pay attention to the base of the number system in which we work. Since we bundle value into digits according to the base we are working in, we carry and borrow according to the value of the base.

REVIEW PROBLEMS

9.11 Perform the indicated addition.

(a) $12_9 + 42_9$

(b) $34_5 + 411_5$

(c) $101110_2 + 1001_2 + 11011_2$

9.12 Perform the indicated subtraction.

(a) $5144_6 - 1023_6$

(b) $713_{12} - A9_{12}$

9.13 Perform the indicated multiplication.

(a) $21_7 \cdot 54_7$

(b) $2102_3 \cdot 121_3$

9.14 Perform the indicated division.

(a) $205_6 \div 15_6$

(b) $1510_8/52_8$

9.15 Determine which of the following are multiples of 5.

(a) 117_9

(b) 111101_2

(c) 111101_4

(d) 4105_6

(e) $A1BA_{15}$

9.16 Determine which of the following are multiples of 12.

(a) 717_8

(b) 2120213_3

(c) 14202_5

(d) 6234_7

(e) $C10B_{13}$

Challenge Problems

9.17 Find the value of the base b such that the following addition problem is correct:

$$\begin{array}{r} 6651_b \\ +\quad 115_b \\ \hline 10066_b \end{array}$$

9.18 Compute: $(10101_2 + 1011_2) \cdot (110011_2 + 1101_2) \div (1000_2 + 100_2 + 10_2 + 1_2 + 1_2)$. **Hints:** 82

9.19 Find the positive base b in which the equation $4 \cdot 12 = 103$ is valid. **Hints:** 141

9.20 Is there any base b for which 3443_b is prime? If so, provide an example. If not, explain why not. **Hints:** 35

9.21 Find the largest prime number (in decimal form) that divides the sum,

$$1_2 + 10_2 + 100_2 + \cdots + 100000000_2.$$

Hints: 125

9.22 A binary number consists of 17 digits, all of which are 1. Triple the number.

(a) How many digits does the new binary number have?

(b) How many of those digits are 1's?

Hints: 153, 78

9.23 Let the product (12)(15)(16), each factor written in base b, equal 3146 in base b. Let $s = 12 + 15 + 16$, each term expressed in base b. Find the value of s in base b. *(Source: AHSME)* **Hints:** 51

9.24 Cleo needs to determine if a 71777-digit base 12 integer is a multiple of 3. The last five digits of the integer are 71777. Find a method that helps Cleo quickly determine whether or not the enormous integer is a multiple of 3. **Hints:** 27

9.25 The evil villain Harris Pilton wrote three secret two-digit numbers, x, y, and z on a napkin. Berris Fueler must name three numbers, A, B, and C, after which Harris will announce the value of $Ax + By + Cz$. If Berris can then name Harris' three secret numbers, Harris will let Berris go. Save Berris! Come up with a way to choose A, B, and C such that Berris is sure to escape Harris' evil clutches. **Hints:** 96

9.26 Find the sum of all the natural numbers that are three-digit palindromes when expressed in base 5. Express your answer in base 5. **Hints:** 149

Extra! *A stupid man's report of what a clever man says is never accurate because he unconsciously translates what he hears into something he can understand.* – Bertrand Russell