CAUCHY INTEGRAL FORMULA AND POWER SERIES

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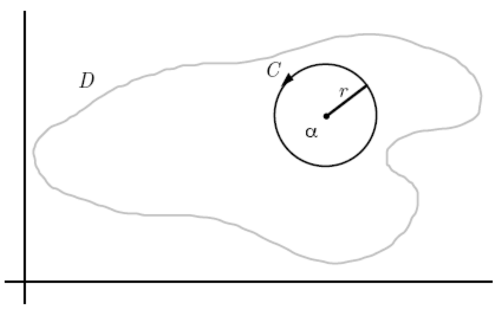
1. Cauchy's Integral Formula

The Cauchy Integral Formula is one of the most powerful theorems in complex analysis. With it we can prove many interesting results regarding anlalytic complex functions. Probably the most powerful being that every differentiable function is equal to it's Taylor Series expansion on it's domain of analyticity. I have already proven many of these theorem in class. However, since the proofs are not all in your book, I thought it might be wise to type them up for you.

1.1. **Cauchy's Integral Formula.**

Theorem 1 (Cauchy's Integral Formula)**.** *Let f be analytic on an open and connected domain $D \subset \mathbb{C}$. Then for any $z \in D$, and any simple closed rectifiable curve $C \subset D$ which wraps in a counterclockwise direction around z,*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw.$$



The integral about the boundary of D

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Proof 1. *First, consider any circle of radius r around z, $C(r,z)$. By the generalized form of Cauchy's Theorem, we know that*

$$\int_{\partial D} \frac{f(w)}{w-z}dw - \int_{C(r,z)} \frac{f(w)}{w-z}dw = 0.$$

So, it suffices to show that

$$\int_{C(r,z)} \frac{f(w)}{w-z}dw = 2\pi i f(z).$$

For the sake of simplifying notation, we will denote $C(r,z) = C' = \{w - z = re^{it} | t \in [0, 2\pi]\}$ Let us note that

$$\begin{aligned}
(1)\quad \int_{C'} \frac{f(w)}{w-z}dw &= \int_{C'} \frac{f(w)-f(z)+f(z)}{w-z}dw \\
(2)\quad &= \int_{C'} \frac{f(w)-f(z)}{w-z}dw + \int_{C'} \frac{f(z)}{w-z}dw \\
(3)\quad &= \int_{C'} \frac{f(w)-f(z)}{w-z}dw + f(z)\int_{C'} \frac{1}{w-z}dw \\
(4)\quad &= \int_{C'} \frac{f(w)-f(z)}{w-z}dw + f(z)\int_{0}^{2\pi} \frac{ire^{it}}{re^{it}}dt \\
(5)\quad &= \int_{C'} \frac{f(w)-f(z)}{w-z}dw + f(z)\int_{0}^{2\pi} idt \\
(6)\quad &= \int_{C'} \frac{f(w)-f(z)}{w-z}dw + f(z)2\pi i
\end{aligned}$$

Since f is continuous on then entire domain with boundary, then it is uniformly continuous. By definition, for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|z - w| < \delta \Rightarrow |f(z) - f(w)| < \epsilon.$$

If we let $r < \delta$ then the left part of equation(6) becomes

$$\begin{aligned}
(7)\quad |\int_{C'} \frac{f(w)-f(z)}{w-z}dw| &\leq \int_{C'} \frac{\epsilon}{\delta}|dw| \\
(8)\quad &\leq \frac{\epsilon}{\delta}\int_{C'} |dw| \\
(9)\quad &\leq \frac{\epsilon}{\delta}2\pi\delta \\
(10)\quad &\leq 2\pi\epsilon
\end{aligned}$$

Since this part of the equation is less than any arbitrary $\epsilon > 0$, then it must be zero. Hence, our proof is complete.

This is an amazing result because it illustrates the relationship between the values of an analytic function on the boundary of a simply connected domain and the values on the interior. The Cauchy Integral Formula states that the values on the interior of a region are uniquely determined by it's values on the boundary.

We can generalize this theorem to finding the values of all of the derivatives of f at z within D by the values of f on the boundary of D. We will do this by induction.

1.2. **Cauchy's Integral Formula for the derivative.** We will now show if all of the above conditions on f and C are satisfied, then

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}dw$$

By definition of the derivative,

$$\begin{aligned}
f'(z) &= \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}\\
&= \lim_{\Delta z\to 0}\frac{1}{\Delta z}[\frac{1}{2\pi i}\int_C \frac{f(w)}{w-(z+\Delta z)}dw - \frac{1}{2\pi i}\int_C\frac{f(w)}{w-z}dw]\\
&= \lim_{\Delta z\to 0}\frac{1}{2\pi i\Delta z}\int_C \frac{f(w)(w-z)-f(w)(w-z-\Delta z)}{(w-z-\Delta z)(w-z)}dw\\
&= \lim_{\Delta z\to 0}\frac{1}{2\pi i\Delta z}\int_C \frac{f(w)\Delta z}{(w-z-\Delta z)(w-z)}dw\\
&= \lim_{\Delta z\to 0}\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z-\Delta z)(w-z)}dw
\end{aligned}$$

To show the derivative is as claimed, we must show that given any $\epsilon > 0$ there exists $\Delta z > 0$ such that

$$|f'(z) - \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}dw| < \epsilon$$

Let us look at the right inequality more carefully.

$$\begin{aligned}
|f'(z) - \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}dw| &= |\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z-\Delta z)(w-z)}dw - \int_C\frac{f(w)}{(w-z)^2}dw|\\
&\le |\frac{1}{2\pi i}|\int_C |\frac{f(w)(w-z)^2 - f(w)(w-z-\Delta z)(w-z)}{(w-z)^3(w-z-\Delta z)}||dw|\\
&\le \frac{1}{2\pi}\int_C \left|\frac{f(w)(w^2-2zw+z^2)-f(w)(w^2-2zw-w\Delta z-+z^2+z\Delta z)}{(w-z)^3(w-z-\Delta z)}\right||dw|\\
&\le \frac{1}{2\pi}\int_C \left|\frac{f(w)(\Delta zw-\Delta zz)}{(w-z)^3(w-z-\Delta z)}\right||dw|\\
&\le \frac{|\Delta z|}{2\pi}\int_C |\frac{f(w)(w-z)}{(w-z)^3(w-z-\Delta z)}||dw|\\
&\le \frac{|\Delta z|}{2\pi}\int_C \left|\frac{f(w)}{(w-z)^2(w-z-\Delta z)}\right||dw|
\end{aligned}$$

At this point we need to set a bound for the functions involved. Since $f(w)$ is continuous on the closed set C, then it is bounded. $f(w) \le M$, for some M. Also, let $d = \min\{|z-w| \mid w \in C\}$ and L be the arclength of C. Since C is rectifiable, we know this to be finite. Then

$$|f'(z) - \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}dw| \le \frac{|\Delta z|ML}{2\pi d^2(d-|\Delta z|)}$$

which tends to zero as Δz goes to zero.

In order to see the pattern, we must compute one more case.

1.3. **Cauchy's Integral formula for case n = 2.**

$$
\begin{aligned}
f''(z) &= \lim_{\Delta z \to 0} \frac{f'(z+\Delta z) - f'(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-(z+\Delta z))^2} dw - \int_C \frac{f(w)}{(w-z)^2} dw \\
&= \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z} \int_C \frac{f(w)(w-z)^2 - f(w)(w-(z+\Delta z))^2}{(w-z-\Delta z)^2 (w-z)^2} dw \\
&= \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z} \int_C \frac{f(w)(w^2-2wz+z^2) - f(w)(w^2-2w(z+\Delta z)+(z+\Delta z)^2)}{(w-z-\Delta z)^2 (w-z)^2} dw \\
&= \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z} \int_C \frac{f(w)(w^2-2wz+z^2-w^2+2wz+2w\Delta z - z^2 - 2z\Delta z - (\Delta z)^2)}{(w-z-\Delta z)^2 (w-z)^2} dw \\
&= \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z} \int_C \frac{f(w)(2w\Delta z - 2z\Delta z - (\Delta z)^2)}{(w-z-\Delta z)^2 (w-z)^2} dw \\
&= \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z} \int_C \frac{f(w)\Delta z(2w-2z-\Delta z)}{(w-z-\Delta z)^2 (w-z)^2} dw \\
&= \lim_{\Delta z \to 0} \frac{1}{2\pi i} \int_C \frac{f(w)(2w-2z-\Delta z)}{(w-z-\Delta z)^2 (w-z)^2} dw \\
&= \frac{1}{2\pi i} \int_C \frac{f(w)(2w-2z)}{(w-z)^2 (w-z)^2} dw \\
&= \frac{2}{2\pi i} \int_C \frac{f(w)}{(w-z)^3} dw
\end{aligned}
$$

1.4. **The Induction Step.** Assume

$$
f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_C \frac{f(w)}{(w-z)^n} dw
$$

Then

$$
\begin{aligned}
f^{(n)}(z) &= \lim_{\Delta z \to 0} \frac{(n-1)!}{2\pi i \Delta z} \int_C \frac{f(w)}{((w-z)-\Delta z)^n} - \frac{f(w)}{(w-z)^n} dw \\
&= \lim_{\Delta z \to 0} \frac{(n-1)!}{2\pi i \Delta z} \int_C \frac{f(w)[(w-z)^n - ((w-z)-\Delta z)^n]}{((w-z)-\Delta z)^n (w-z)^n} dw \\
&= \lim_{\Delta z \to 0} \frac{(n-1)!}{2\pi i \Delta z} \int_C \frac{f(w)[(w-z)^n - ((w-z)^n + n(w-z)^{n-1}(-\Delta z) + \cdots + n(w-z)(-\Delta z)^{n-1} + (}{((w-z)+\Delta z)^n (w-z)^n} \\
&= \lim_{\Delta z \to 0} \frac{(n-1)!}{2\pi i \Delta z} \int_C \frac{f(w)\Delta z(n(w-z)^{n-1} + \cdots + n(w-z)(-\Delta z)^{n-2} + (-\Delta z)^{n-1})}{((w-z)+\Delta z)^n (w-z)^n} dw \\
&= \lim_{\Delta z \to 0} \frac{(n-1)!}{2\pi i} \int_C \frac{f(w)(n(w-z)^{n-1} + \cdots + n(w-z)(-\Delta z)^{n-2} + (-\Delta z)^{n-1})}{((w-z)+\Delta z)^n (w-z)^n} dw \\
&= \frac{(n-1)!}{2\pi i} \int_C \frac{f(w)n(w-z)^{n-1}}{(w-z)^n (w-z)^n} dw \\
&= \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}} dw
\end{aligned}
$$

By induction, we have the general form of Cauchy's Integral Formula.

Theorem 2. *Let f be analytic on a simply connected domain D, and continuous on the rectifiable boundary C. Then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}dw$$

2. Power Series

The general form of Cauchy's Integral Formula allows us to see that if a complex function is differentiable on a domain, then it is smooth (or infinitely differentiable) on the domain. In fact, the formula proves that the value of an analytic function and all of it's derivatives is uniquely determined by the value of the function on the boundary. We will now prove that a complex differentiable is equal to it's Taylor series expansion on it's domain of differentiability.

First note that,

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}}$$

Analyzing the geometric series, we see that

$$(1+\alpha+\alpha^2+\cdots+\alpha^{n-1})(1-\alpha) = 1-\alpha^n$$
$$(1+\alpha+\alpha^2+\cdots+\alpha^{n-1}) = \frac{1-\alpha^n}{1-\alpha}$$
$$(1+\alpha+\alpha^2+\cdots+\alpha^{n-1}) = \frac{1}{1-\alpha} - \frac{\alpha^{n-1}}{1-\alpha}$$
$$(1+\alpha+\alpha^2+\cdots+\alpha^{n-1}) + \frac{\alpha^n}{1-\alpha} = \frac{1}{1-\alpha}$$

Therefore, by letting $\alpha = \frac{z-z_0}{w-z_0}$, we achieve the following result,

$$\frac{1}{w-z} = \frac{1}{w-z_0}(1+\frac{z-z_0}{w-z_0}+(\frac{z-z_0}{w-z_0})^2+\cdots(\frac{z-z_0}{w-z_0})^{n-1}+\frac{(\frac{z-z_0}{w-z_0})^n}{1-\frac{z-z_0}{w-z_0}})$$

$$\frac{f(w)}{(w-z)} = \frac{f(w)}{w-z_0}(1+\frac{z-z_0}{w-z_0}+(\frac{z-z_0}{w-z_0})^2+\cdots(\frac{z-z_0}{w-z_0})^{n-1}+\frac{(\frac{z-z_0}{w-z_0})^n}{1-\frac{z-z_0}{w-z_0}})$$
$$= \frac{f(w)}{w-z_0}+\frac{f(w)(z-z_0)}{(w-z_0)^2}+\cdots\frac{f(w)(z-z_0)^{n-1}}{(w-z_0)^n}+\frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}-(z-z_0)(w-z_0)^n}$$
$$= \frac{f(w)}{w-z_0}+\frac{f(w)(z-z_0)}{(w-z_0)^2}+\cdots\frac{f(w)(z-z_0)^{n-1}}{(w-z_0)^n}+\frac{f(w)(z-z_0)^n}{(w-z_0)^n(w-z_0-z+z_0)}$$
$$= \frac{f(w)}{w-z_0}+\frac{f(w)(z-z_0)}{(w-z_0)^2}+\cdots\frac{f(w)(z-z_0)^{n-1}}{(w-z_0)^n}+\frac{f(w)(z-z_0)^n}{(w-z_0)^n(w-z)}$$
$$= \frac{f(w)}{w-z_0}+\frac{f(w)}{(w-z_0)^2}(z-z_0)+\cdots\frac{f(w)}{(w-z_0)^n}(z-z_0)^{n-1}+\frac{f(w)}{(w-z_0)^n(w-z)}(z-z_0)^n$$

Now we can integrate each term with respect to w,

$$\int_C\frac{f(w)}{(w-z)}dw = \int_C\frac{f(w)}{w-z_0}dw+\cdots\int_C\frac{f(w)}{(w-z_0)^n}dw(z-z_0)^{n-1}+\int_C\frac{f(w)}{(w-z_0)^n(w-z)}dw(z-z_0)^n$$

By the generalized cauchy integral formula, we know that

$$2\pi i\frac{f^{(n)}(z_0)}{n!} = \int_C\frac{f(w)}{(w-z_0)^{n+1}}dw$$

and, therefore, get

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \cdots \frac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + R_n$$

where

$$R_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^n (w - z)} dw (z - z_0)^n$$

We need to show that $|R_n| \to 0$ as $n \to \infty$. In order to do this, we need to realize that f is continuous on C and, therefore, have a maximum value on C. Let $f(w) \leq M$ for $w \in C$. Without loss of generality, we can assume C is a circle of radius r about z_0. Therefore, $|w - z_0| = r$ for all $w \in C$. If we let $|z - z_0| = d$, then

$$r - d \leq |w - z_0| - |z_0 - z| \leq |w - z + z + z_0| \leq |w - z|$$

implies

$$\frac{1}{r - d} \geq \frac{1}{|w - z|}$$

$$\begin{aligned} |R_n| &\leq |\frac{1}{2\pi i}| \int_C \frac{|f(w)|}{|w - z_0|^n (w - z)} |dw| (|z - z_0|)^n \\ &\leq \frac{1}{2\pi} \int_C \frac{M}{r^n (r - d)} d^n |dw| \\ &\leq \frac{M d^n}{2\pi r^n (r - d)} \int_C |dw| \\ &\leq \frac{M d^n}{2\pi r^n (r - d)} 2\pi r \\ &\leq \frac{M r}{r - d} (\frac{d}{r})^n \end{aligned}$$

Since z is within the C, then $d < r$ and $(\frac{d}{r})^n \to 0$ as $n \to \infty$. Therefore, $f(z)$ converges to it's Taylor series expansion everywhere it is differentiable.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

This statement has further and more amazing consequences. Consider the following concept.

Definition 1. If $f(z)$ is a nonconstant analytic function in a domain D, and $f(a) = 0$, then we define a to be a zero of order h if $h = min\{n \in \mathbb{N} \mid f^{(n)}(a) \neq 0\}$.

How do we know such an h exists?
Let us assume no such h exists. Then $f^{(n)}(a) = 0$ for all n. Then for all since f is analytic in D, then $f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z - a)^n$ for all $z \in B(r, a) \subset D$, and for some r. This, however, would imply that f is constant and $f(z) \equiv 0$ in $B(r, a)$. Now we need to show that f is identically zero on all of D. This proof requires some topology. For those who are unfamiliar with this concept, D is connected if and only if the only subsets of D which are both open and closed, are D itself, and the empty set. Let $E = \{z \in D \mid f^{(n)}(z) = 0, \forall n\}$. Since f

is continuous, then the pre-image of closed sets are closed, so E is a closed set. However, we have just shown that if $a \in E$ then there exists $r > 0$ such that $B(r, a) \subset E$, which implies E is open. Therefore $E = \emptyset$ or $E = D$. Hence if there exists an $a \in E$, then $E = D$. We know have the following theorem:

Theorem 3. *If f is analytic on a connected domain, D, and there exists $a \in D$ such that $f^{(n)}(a) = 0$ for all $n \in \mathbb{N}$, then $f \equiv 0$ for all $z \in D$.*

or equivalently,

Theorem 4. *If f is a nonconstant analytic on a connected domain, D, and $f(a) = 0$, then*

$$f(z) = (z - a)^n f_n(z)$$

for some $n \in \mathbb{N}$, and some analytic function $f_n(z)$ such that $f_n(a) \neq 0$. We call n the order of the zero, a.

It is necessary to point out that f is analytic, and more generally, it is continuous. Therefore, if $f_n(a) \neq 0$, there must exist $r > 0$ such that $f_n(z) \neq 0$ for all $z \in B(r, a)$.

This tells us that an analytic function acts locally like a polynomial. Which leads us to the following two results.

Corollary 1 (Identity Theorem)**.** *The zeros of an analytic function are discrete. Let f be a nonconstant analytic function on a connected domain D.*
If there exists $r > 0$ such that $f \equiv 0$ for all $z \in B(r, a)$, then $f \equiv 0$ for all $z \in D$.

3. Homework

These problems use Liousville's Theorem and the Maximum-Modulus Theorem;

Problem 1. Show that if $P(z)$ is a nonconstant polynomial on a domain D, then it must have at least one zero.
(Hint: Show that if it does not have a zero, then $\dfrac{1}{P(z)}$ is entire and bounded.)

Problem 2. Show that if f is entire and $Re(f(z)) \leq M$ for all z and some $M \in \mathbb{R}$, then f is a constant function.
(Hint: Consider e^f)

Problem 3. Show that if f is analytic and f is nonconstant in a region D, and $f(z) \neq 0$ for all $z \in D$, then $|f|$ does not have a minimum in D.

Problem 4. Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic on a region D. Prove that if $u(x, y)$ is bounded, then u must be constant.

This problem uses the Identity Theorem.

Problem 5 (Uniqueness Theorem)**.** If f and g are analytic on a domain D, and $f \equiv g$ on a $B(r, a) \subset D$, for some $r > 0$ and $a \in D$, then $f \equiv g$ for all $z \in D$.

4. Sources

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