Complex Functions

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September 15, 2015

Complex Functions

Let $f : \mathbb{C} \to \mathbb{C}$ be a complex valued function on $\mathbb{C}$.
Let $z = x + iy$.
Then we can write $f(z) = w$ or equivalently $f(x + iy) = w(x + iy)$.
Since w has a real an imaginary part which depend on the real and imaginary part of z, then, necessarily, there exists two functions

$$u : \mathbb{R}^2 \to \mathbb{R} \text{ and } v : \mathbb{R}^2 \to \mathbb{R}$$

such that

$$f(x + iy) = u(x, y) + iv(x, y)$$

Note that u and v are multi-variable real valued functions giving us the real and imaginary part of $f(z)$, respectively.

Example A

Let $f(z) = \dfrac{1}{z}$.
Since $f(z) = \dfrac{\bar{z}}{|z|^2}$, then $f(x+iy) = \dfrac{x-iy}{x^2+y^2}$.
Breaking up the fraction into real and imaginary parts, we get,

$$f(x+iy) = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$

So $u(x,y) = \dfrac{x}{x^2+y^2}$ and $v(x,y) = \dfrac{-y}{x^2+y^2}$. Clearly, $x^2+y^2 \neq 0$ if and only if $x \neq 0 \wedge y \neq 0$ if and only if $z \neq 0$.

Real part of $\frac{1}{z}$

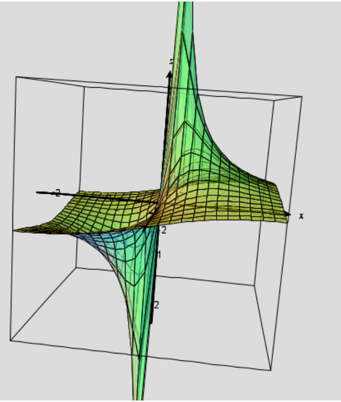

Figure: $u(x, y) = \frac{x}{x^2 + y^2}$

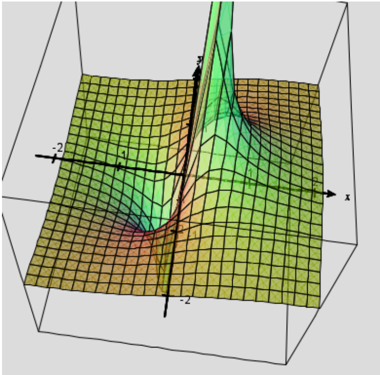

Figure: $u(x, y) = \frac{x}{x^2 + y^2}$

Imaginary part of $\frac{1}{z}$

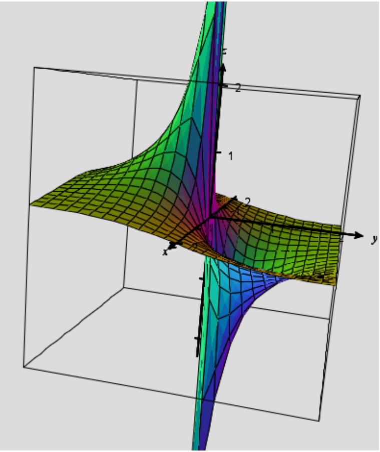

Figure: $v(x, y) = \frac{-y}{x^2 + y^2}$

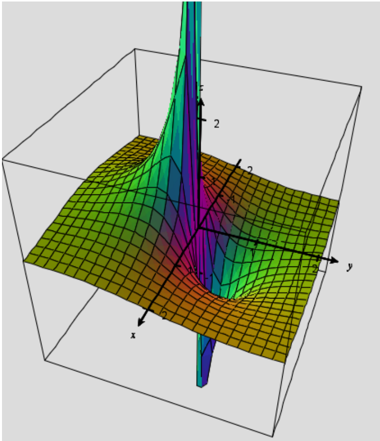

Figure: $v(x, y) = \frac{-y}{x^2 + y^2}$

Example B

Let $g(z) = z^2 - 3z + 2$.
If $z = x + iy$, then

$$\begin{aligned} g(x+iy) &= (x+iy)^2 - 3(x+iy) + 2 \\ &= x^2 + 2ixy - y^2 - 3x - 3iy + 2 \end{aligned}$$

Then $u =$
and $v =$
What does the graph of the real and imaginary parts look like?

Example C

Let $h(z) = z^2 - 1$.
Find the real and imaginary parts.
Graph them.

Limits

What does it mean for f to be continuous?

$$\lim_{z \to z_o} f(z) = w_o$$

For any $\epsilon > 0$,
there exists $\delta > 0$
such that

$$|z - z_o| < \delta \text{ implies } |f(z) - w_o| < \epsilon$$

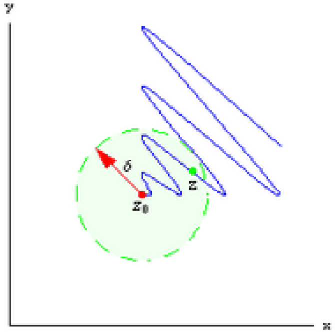

Figure: Complex Limit can come from any direction

Limits on a plane

Recall, the set of complex numbers forms a plane, and therefore, for z can approach z_o from any direction within $B_{z_o,\delta}$. The image of z, $f(z)$, is within the ball, $B_{w_o,\epsilon}$.

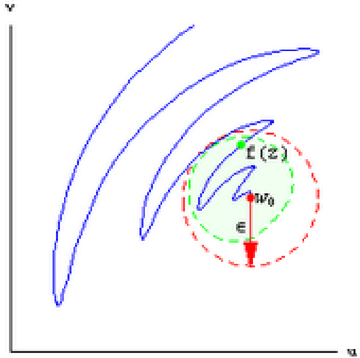

Figure: Continuity in the complex plane

There is a square about z_o, given by the set

$$\{x+iy|\,|x-x_o|<\frac{\epsilon}{2},|y-y_o|<\frac{\epsilon}{2}\}$$

which is contained completely within the Ball of radius delta about z_o. Conversely, for any square about z_o, there is a disc contained completely in the square. Therefore, geometrically, both metrics, or methods of measuring distance seem to agree.

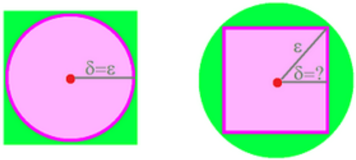

Figure: Find the lengths of the sides of the square for each picture

Limits in two real dimensions

Analytically, we can see if $z = x + iy$ and $z_o = a + ib$ then

$$|(x+iy)-(a+ib)| = |(x-a)+i(y-b)|$$

The expression to the right is only zero when $x - a$ and $y - b$ are zero. Therefore,

$$\lim_{z \to z_0} f(z) = f(z_0)$$

is equivalent to

$$\lim_{\substack{x \to x_o \\ y \to y_o}} f(x + iy) = w_o$$

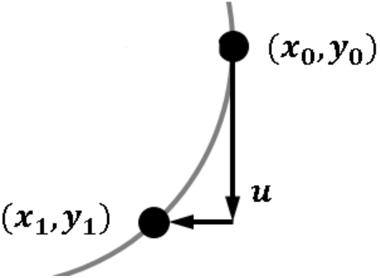

Figure: The limit in the x and y direction would give the same answer as in any direction

More Limits

If we let $w_o = c + id$, then we get

$$\lim_{\substack{x \to x_o \\ y \to y_o}} u(x, y) + iv(x, y) = c + id$$

or

$$\lim_{\substack{x \to x_o \\ y \to y_o}} u(x, y) = c$$

and

$$\lim_{\substack{x \to x_o \\ y \to y_o}} v(x, y) = d$$

All of the algebraic properties of limits in one real variable calculus hold in single complex variables.

Limit Properties

Write down the four limit properties below.

Example

Let $f(z) = \dfrac{z^2 - 4}{z + 2}$. Prove that

$$\lim_{z \to -2} f(z) = -4$$

Solution

Let $\epsilon > o$.
Since $f(z) = z - 2$ for all $z \neq -2$, then the following are equivalent:

$$\begin{aligned} |f(z) - -4| &< \epsilon \\ |f(z) + 4| &< \epsilon \\ |z - 2 + 4| &< \epsilon \\ |z + 2| &< \epsilon \end{aligned}$$

So if $\delta = \epsilon$ then $|z + 2| < \epsilon$ implies $|f(z) + 4| < \epsilon$.

What is e^z?
It is e^{x+iy}. What is the image of a vertical line?
What is the image of a horizontal line?
What is the image of any line through the origin?

e^z

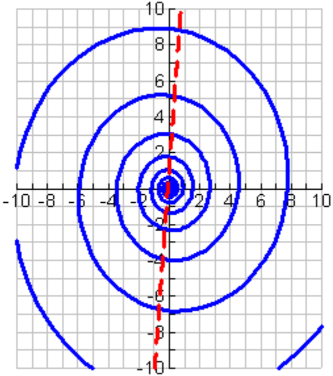

Figure: The image under the exponential functions of a line with positive slope

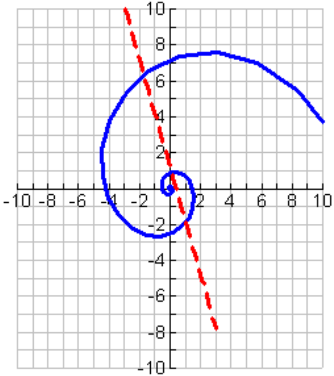

Figure: The image under the exponential functions of a line with negative slope

Real and Imaginary Parts

Homework:
Separate into the real and imaginary parts.

- $\sin(z)$
- $\cos(z)$
- e^z

Limits

Homework: 2.89 Prove that

$$\lim_{z \to i}(z^2 + 2z) = 2i - 1$$

Limits

Homework: 2.94 Evaluate

$$\lim_{z \to 2i} iz^4 + 3z^2 - 10i$$

$$\lim_{z \to e^{\frac{\pi i}{3}}} (z - e^{\frac{\pi i}{3}})(\frac{z}{z^3 + 1})$$