Complex Polar Derivatives

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Polar Coordinates

Now that we know that the Cauchy-Riemann Equations are a sufficient condition for differentiability, we can see how this affects our interpretation of complex functions in polar form. We have already seen through our study of complex numbers, that polar form is more helpful when considering complex division and multiplication. Could it be more helpful in understanding complex functions?

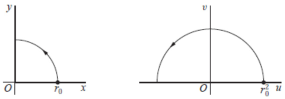

FIGURE 19
$w = z^2$.

Figure: The mapping of the quarter unit circle under z^2

Complex Polar: When $z \neq 0$

For all of this section $z \neq 0$. Let $x = r\cos(\theta)$ and $y = r\sin(\theta)$.
Suppose $w = f(z)$.
Then in substituting $z = x + iy$, we can write, $f(x + iy) = f(r, \theta)$.
Then by chain rule, we get that

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

. On your white boards, write chain rule for θ

u_r and u_θ

Using $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we get

$$u_r = u_x \cos(\theta) + u_y \sin(\theta)$$

. What do you get for u_θ?

v_r and v_θ

We can use the same idea for the imaginary part of f to get

$$v_\theta = -v_x r \sin(\theta) + v_y r \cos(\theta)$$

. What do you get for v_r?

Using Cauchy-Riemann Equations

If we assume our function f is differentiable, then

$$u_x = v_y$$

and

$$u_y = -v_x$$

Use this and our calculations for v_r, v_θ, u_r and u_θ to get Cauchy Riemann Eqautions in polar.

Example

Let us begin with u_r.

$$\begin{aligned} u_r &= u_x \cos(\theta) + u_y \sin(\theta) \\ &= v_y \cos(\theta) - v_x \sin(\theta) \\ &= \frac{1}{r} v_\theta \end{aligned}$$

So $ru_r = v_\theta$. You try to find a relationship between u_θ and v_r.

$u_x and u_y$

Using the formulas for u_r and u_θ, compute

$$u_r \cos(\theta) - \frac{1}{r} u_\theta \sin(\theta)$$

and

$$u_r \sin(\theta) + \frac{1}{r} u_\theta \cos(\theta)$$

$f'(z)$

If the C-R equations hold, then we already know that

$$f'(z) = u_x + iv_x$$

Use the information on the previous slide to find $f'(z)$ in terms of the partials with respect to r and $e^{-i\theta}$.

Theorem

Let the function $f(z) = u(r, \theta) + iv(r, \theta)$ be defined throughout some ϵ neigborhood of a nonzero point $z_0 = r_0 e^{i\theta_0}$, $B_\epsilon(z_0)$.
Suppose that the first partial derivatives $u_r, u_\theta, v_r, v_\theta$ exist everywhere in the neighborhood.
The following are equivalent:

- $ru_r = v_\theta$ and $u_\theta = -rv_r$
- $u_x = v_y$ and $u_y = v_x$
- $f'(z)$ exists
- $f'(z) = u_x + iv_x$
- $f'(z) = e^{-i\theta}(u_r + iv_r)$

Problem

Let $f(z) = \dfrac{\bar{z}}{|z|}$ and $g(z) = \dfrac{\bar{z}}{|z|^2}$.

Find the domain of differentiability of f and g using polar coordinates.