Conformal Maps and Applications

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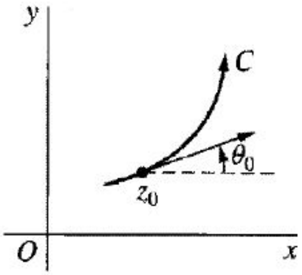
Image of a Curve

Suppose C is a smooth curve passing through z_0. In other words, $r(t_0) = z_0$ for some t_0, where $r(t)$ is a parametric representation of C.

Let $w(t) = f(r(t))$ be the image, Γ, of the curve, C, under an analytic function f.

According to chain rule,

$$w'(t) = f'(r(t))r'(t)$$



Angle of Inclination

Now let us consider the change in angle of the tangent vectors at z_0. Since the argument of the product of complex numbers is the sum of the arguments, $e^{a+b} = e^a e^b$, then we can say that

$$\arg w'(t) = \arg \frac{df(r(t))}{dt} = \arg f'(r(t)) + \arg r'(t)$$

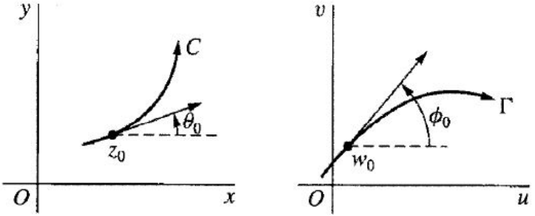


FIGURE 129
$\phi_0 = \psi_0 + \theta_0$.

Angle of Inclination

The angle of inclination of a line tangent to C at $z_0 = r(t_0)$ is any value θ_0 of $\arg(r'(t))$.
If $\Psi_o = \arg f'(z_0)$, then $\Phi_0 = \Psi_0 + \theta_0$,
where $\Phi_0 = \arg(w'(t_0)) =$ angle $of \Gamma'$.

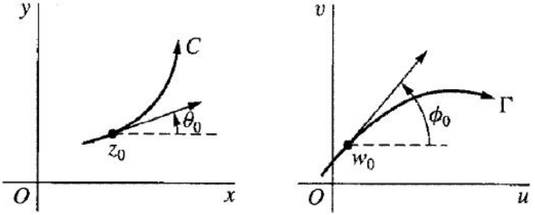
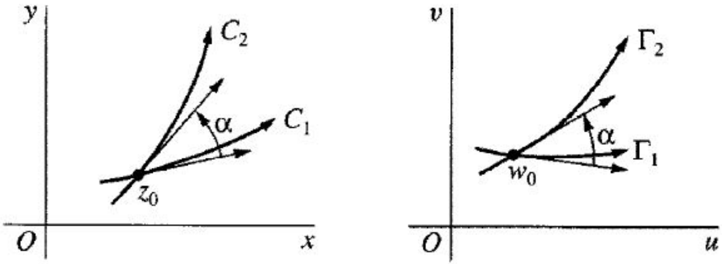


FIGURE 129
$\phi_0 = \psi_0 + \theta_0$.

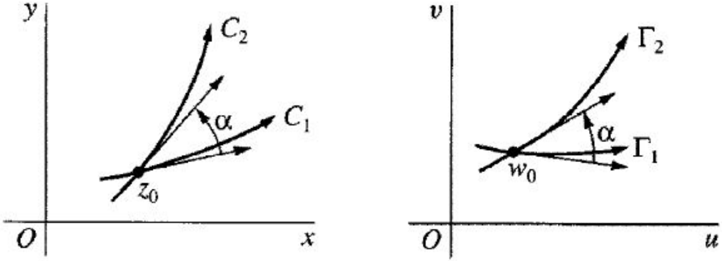
Angle of Inclination

Letting C_1 and C_2 be two smooth curves passing through z_0 in the $x + iy$ plane.
Let f be an analytic function and suppose $f(x, y) = u(x, y) + iv(x, y)$.



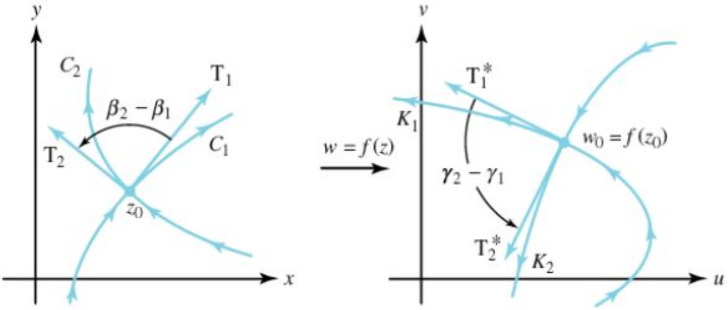
Angle of Inclination

Let $f(C_1) = \Gamma_1$ and $f(C_2) = \Gamma_2$.
Let $z_0 \in C_1 \cap C_2$ and $w_0 = \Gamma_1 \cap \Gamma_2$.



Conformal Mappings

Let C_1 and C_2 be two smooth arcs passing through z_0.
Let θ_1 and θ_2 are the angles of inclination of the directed lines tangent to C_1 and C_2, respectively at z_0.
The angle between C_1 and C_2 is $\theta_2 - \theta_1$.

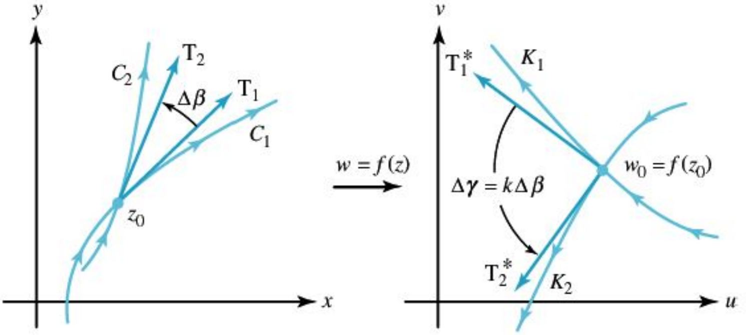


Conformal Mappings

Then the angles of inclination of the directed lines tangent to the image curves provide the differences,
$\arg(w_k(t)) = \arg(f(z_0) + \arg(r_k(t))$.

$$\Phi_1 = \Psi_0 + \theta_1 \qquad \Phi_2 = \Psi_0 + \theta_2$$

where $\Phi_o = \arg f'(z_0)$ and $\theta_0 = \arg(r_k'(t))$.

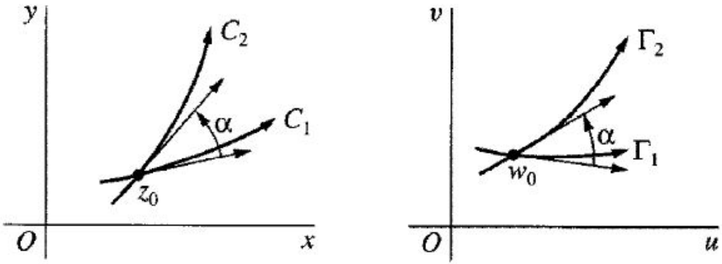


Conformal Mappings

$$\Phi_1 = \Psi_0 + \theta_1 \qquad \Phi_2 = \Psi_0 + \theta_2$$

To find the angle between Γ_1 and Γ_2, we need to

$$\Phi_2 - \Phi_1 = \Psi_0 + \theta_2 - \Psi_0 - \theta_1 \quad (1)$$
$$= \theta_2 - \theta_1 \quad (2)$$
$$= \arg(C_1, C_2) \quad (3)$$



Conformal Mappings

In other words, if $f(z)$ is analytic, and $f'(z) \neq 0$, then $f(z)$ is angle preserving, or conformal.
Therefore, a conformal mapping will denote an analytic mapping whose derivative is never zero.

Collaries and Properties of Conformal Mappings

- If $C_1 \perp C_2$ implies $\Gamma_1 \perp \Gamma_2$.
- All conformal mappings have local inverses
- If $f = u + iv$ is analytic, u is the harmonic conjugate of v.

Transformation of Harmonic Functions

Problem

To find a harmonic function on a given domain satisfying certain conditions on the boundary of the domain.

- Values of the function are prescribed along the boundary - Boundary Value Problem of the 1st Kind (Dirichlet Problem)
- Values of normal derivative of the function are prescribed on the boundary - Boundary Value Problem of 2nd Kind (Newmann Problem)

Example

Example

Let $f(z) = -ie^{iz}$.

$$
\begin{aligned}
f(z) &= -ie^{iz} && (4)\\
&= -ie^{i(x+iy)} && (5)\\
&= -ie^{ix-y} && (6)\\
&= -ie^{ix}e^{-y} && (7)\\
&= -i(\cos(x) + i\sin(x))e^{-y} && (8)\\
&= e^{-y}\sin(x) - ie^{-y}\cos(x) && (9)
\end{aligned}
$$

Still the same example

$$u(x, y) = e^{-y} \sin(x) \qquad v(x, y) = -e^{-y} \cos(x)$$

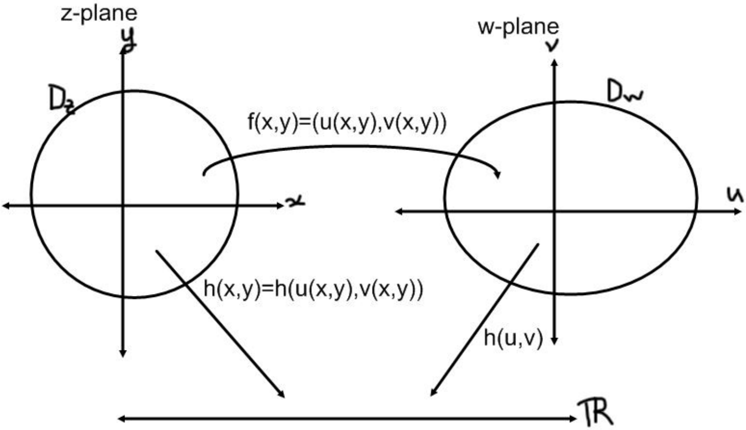
Since $f(z)$ is analytic, both u and v are harmonic.
We also have boundary conditions:

$$u(0, y) = 0 \qquad u(\pi, y) = 0 \tag{10}$$

$$u(x, 0) = \sin(x) \qquad \lim_{y \to \infty} u(x, y) = 0 \tag{11}$$

Theorem 1

Let $f(x, y) = u(x, y) + iv(x, y)$ maps a region, D_z, to D_w.
If h is harmonic in D_w, then
$H(x, y) = h(u(x, y), v(x, y)) = h \circ f(x, y)$.



Theorem 2

Suppose that the analytic function

$$f(z) = u(x, y) + iv(x, y)$$

maps an arc C in the z-plane onto an arc Γ is w-plane.
Let $f(z)$ be conformal on C and let $h(u, v)$ be differentiable on Γ.
If $h(u, v)$ satisfies

$$h = c$$

or

$$\frac{dh}{dn} = 0$$

along Γ, then

$$H(x, y) = h(u(x, y), v(x, y))$$

satisfies the corresponding condition along C.

Steady Temperatures

In the theory of heat

$$\Phi = \text{ flux across surface within a solid homogenous body} \tag{12}$$

$$\text{at a point on that surface } \frac{cal}{sec^2 cm^2} \tag{13}$$

$$= \text{ quantity of heat flowing in specified direction} \tag{14}$$

$$\text{normal to surface per unit time/area} \tag{15}$$

$$= -K\frac{dT}{dn} \tag{16}$$

$$= \text{ varies with normal derivatives} \tag{17}$$

where k = thermal conductivity.

Heat Conduction Theory

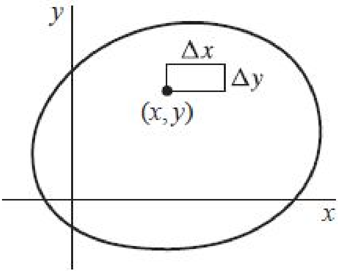
Restrictions

- no heat sources or sinks (thermal energy is created or destroyed)
- temp does not vary with time
- temperature varies only with x, y coordinates

T doesn't vary along with the z-coordinate which means that the heat flow is 2-dimensional and parallel to the xy-plane.

Steady State

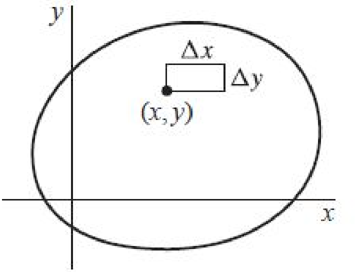
Letting (x, y) be a point in the interior of solid.
A volume element can be thought of as a rectangular prism of unit height, with base Δx by Δy.



Steady State

The time rate of flow of heat toward the left-hand face is
$-KT_x(x, y)\Delta y$
The time rate of flow of heat toward the right-hand face is
$KT_x(x + \Delta x, y)\Delta y$
To obtain the net rate of heat loss from the element through those two faces:

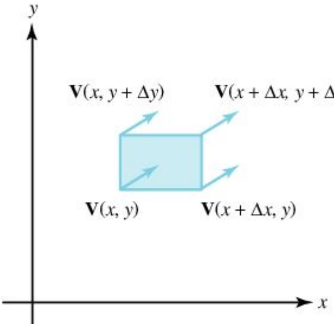
$$-K[\frac{T_x(x + \Delta x, y) - T_x(x, y)}{\Delta x}]\Delta x\Delta y \approx \frac{d^2 T}{dx^2} dxdy$$



Steady State

The time rate of flow of heat toward the top face is
$-KT_y(x, y + \Delta y)\Delta x$
The time rate of flow of heat toward the bottom face is
$KT_y(x, y)\Delta x$
To obtain the net rate of heat loss from the element through these two faces:

$$-K[\frac{T_y(x, y + \Delta y) - T_y(x, y)}{\Delta y}]\Delta x \Delta y \approx \frac{d^2 T}{dy^2} dxdy$$



Steady State

Since the heat can only enter or leave the volume element from one of those four faces, and the temperature within is steady, then

$$\frac{d^2T}{dx^2} + \frac{d^2T}{dy^2}dxdy = 0$$

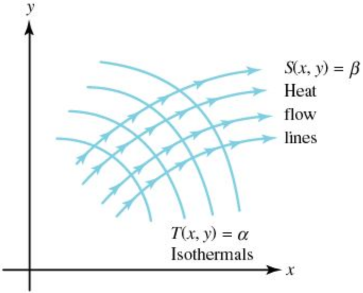
which means $T(x, y)$ is a harmonic function in the interior of the solid body.

isotherms

Level curves of the temperature equation, T, are called *isotherms*.

$$T(x, y) = c$$

is a level curve at c.



(b) Heat flow lines and isothermals.

Gradient of T

We know that the gradient is always perpendicular to a level curve at each point, $\nabla T \perp$ level curve (isotherm).

The gradient of T is always pointing in the direction of maximum increase, so ∇T points in the direction of maximum heat flux.

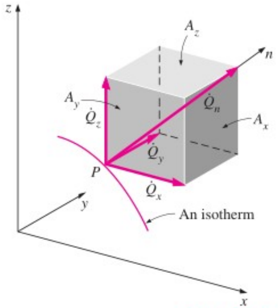


FIGURE 2–8
The heat transfer vector is always normal to an isothermal surface and can be resolved into its components like any other vector.

Harmonic Conjugates

Recall that the Cauchy-Riemann Equations give us that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{18}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{19}$$

This tells us that the tangent vector of $v(x, y)$,

$$(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}) = (-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x})$$

is orthogonal to the tangent vector to $u(x, y)$, $(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$.

Let S be the harmonic conjugate of T, then $S(x, y) = c$ is a curve whose tangent vector is ∇T.

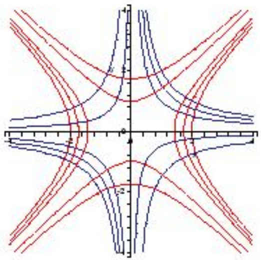
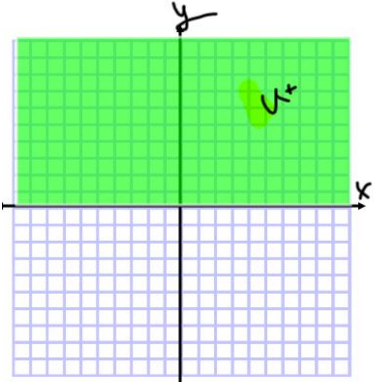


Figure 2: Level curves for two harmonic conjugates

Example: Steady Temperatures in a Half Plane

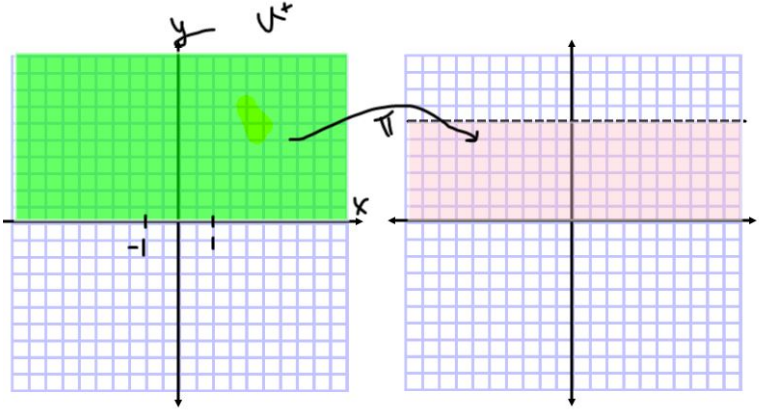
Let $U^+ = \{x + iy \mid y \geq 0\}$ be the upper half plane.
Find a harmonic function, $T(x,y) : U^+ \to \mathbb{R}$, such that

- $T(x,y) \leq M$
- $\lim_{y\to\infty} T(x,y) = 0$
- $T(x,0) = \begin{cases} 0 & \text{for } x > 1 \text{ or } x < -1 \\ 1 & \text{for } -1 \leq x \leq 1 \end{cases}$



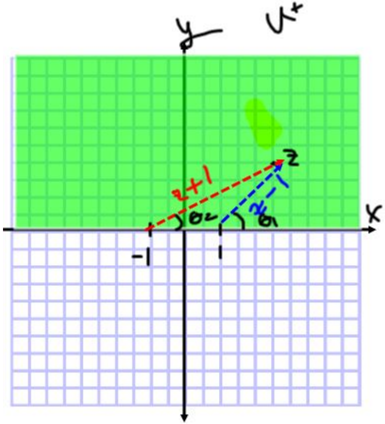
Dirichlet Problem

To solve this problem, it is easier to first transform our upper half plane to the strip $A = \{x + iy \mid 0 \leq y < \pi\}$.



How do we do this?

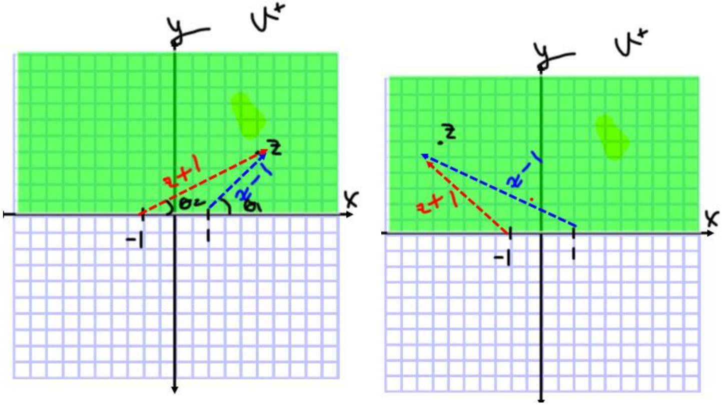
Transformations



If we allow $z-1=r_1e^{i\theta_1}$ and $z+1=r_2e^{i\theta_2}$, then what does $\dfrac{z-1}{z+1}$ do?

Transformations

Try to find the restrictions on $\dfrac{z-1}{z+1}$.



As you can see, both $0 \leq \arg(z+1), \arg(z-1) \leq \pi$, which means $0 \leq \arg(\dfrac{z-1}{z+1}) < \pi$.

The log function $f(z) = \log(\frac{z-1}{z+1}) = \ln(\frac{r_1}{r_2}) + i(\theta_1 - \theta_2)$

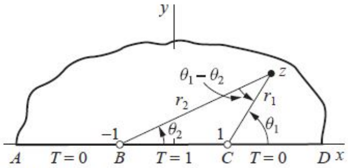


FIGURE 140

What is the image of the interval $[-1, 1] \subset \mathbb{R}$?
What is the image of the intervals $(-\infty, -1) \cup (1, \infty)$?

ALMOST DONE!

So which function $T(u, v)$ in the strip from $A = \{z \in \mathbb{C} \mid 0 \leq Im(z) \leq \pi\}$ satisfies

- T is harmonic on A
- $T(u, v) \leq M$
- $\lim_{v \to \infty} T(u, v) = 0$
- $T(u, 0) = \begin{cases} 0 & \text{for } v = 0 \\ 1 & \text{for } v = \pi \end{cases}$

How about $T(u, v) = \frac{1}{\pi} v$?

Is it the imaginary part of $\frac{w}{\pi}$?

Does it satisfy all of the conditions?

Now all we need to do is compose this function with $w = \log(\frac{z-1}{z+1})$.