FIRST ORDER DIFFERENTIAL EQUATIONS

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1. The Integrating Factor

A first order linear differential equation is the first we encounter which is not separable. It is of the form

$$y' + P(x)y = Q(x).$$

One might notice that the left side of this equation looks very much like the derivative of a product of functions, F, y. If this were the case, it would be

$$\frac{dF \cdot y}{dx} = F'y + Fy'$$

As we always do in these mathematical circumstances, we force the equation to take the form we would like. The problem, is that this F, if it were to exist, is in front of our y. The only way this can happen, is if we multiply every thing in the differential equation by this elusive F.

$$Fy' + FP(x)y = FQ(x)$$

Now the left hand side is supposed to be the derivative of the product Fy. And hence,

$$Fy' + F'y = Fy' + FP(x)y$$

Therefore, we see that

$$F' = FP(x)$$

Now we can solve for F, by using separable differential equations:

$$\begin{aligned} \frac{dF}{dx} &= FP(x) \\ \frac{dF}{F} &= P(x)dx \\ \int \frac{dF}{F} &= \int P(x)dx \\ \ln(F) &= \int P(x)dx \\ F &= e^{\int P(x)dx} \\ F(x) &= e^{\int P(x)dx} \end{aligned}$$

We call F the *integrating factor*. Consequently, the differential equation becomes

$$e^{\int P(x)dx}y' + e^{\int P(x)dx}P(x)y = e^{\int P(x)dx}Q(x)$$

and since the left hand is the derivative of Fy, we can write

$$\frac{d(e^{\int P(x)dx})y}{dx} = e^{\int P(x)dx}Q(x)$$

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Integrating both sides and using the Fundamental Theorem of Calculus, we achieve

$$e^{\int P(x)dx}y = \int Q(x)e^{\int P(x)dx}dx + C$$

Dividing both sides by the integrating factor, we get

$$y = \frac{1}{e^{\int P(x)dx}}\int Q(x)e^{\int P(x)dx}dx + C$$

2. Solving a First Order Differential Equation

Example 1. An object of mass $10kg$ is released from a hot air balloon. Find the distance if falls in 10 seconds, if the force due to air resistance is directly proportional to the speed of the object.

Solution 1. Let g be the gravitational constant, then the force due to gravity will be $100g$.
If v is the velocity of the object, then the force of air resistance is kv. Since air resistance is a force in the opposite direction, the total force will be given be

$$100g - kv$$

Using Newton's Second law of motion, $F = ma$, we achieve the differential equation

$$100a = 100g - kv.$$

Since $a = \frac{dv}{dt}$, we arrive at the first order differential equation

$$100\frac{dv}{dt} = 100g - kv.$$

or

$$kv + 100\frac{dv}{dt} = 100g$$
$$\frac{dv}{dt} - \frac{k}{100}\frac{dv}{dt} = g$$

Let $c = \frac{k}{100}$ to simplify notation. We can see that $y = v$, $P(x) = c$, and $Q(x) = g$.
The integrating factor is then

$$F = e^{\int cdt} = e^{ct}$$

Then by the above formula,

$$v = \frac{1}{e^{ct}}\int ge^{ct}dt$$
$$v = \frac{g}{e^{ct}}\int e^{ct}dt$$

Letting $u = ct$ then $du = cdt$ and consequently $dt = \frac{1}{c}du$.

$$v = \frac{g}{e^{ct}}(\frac{1}{c}e^{ct} + C)$$

Distributing the reciprocal of the integrating factor, we achieve,

$$v = \frac{g}{c} + gCe^{-ct}$$

We know the initial velocity to be 0, so we can solve for C,

$$0 = \frac{g}{c} + gC$$

which yields

$$C = -\frac{1}{c}$$

Hence,

$$v(t) = \frac{g}{c} - \frac{g}{c}e^{-ct}$$

Since distance is the integral of velocity, we must integrate both sides.

$$s(t) = \int vdt$$

which yields,

$$s(t) = \int \frac{g}{c}(1 - e^{-ct}dt).$$

Another u substitution will allow us to integrate:

$$s(t) = \frac{g}{c}(t - \frac{1}{-c}e^{-ct} + D).$$

The initial distance is 0, so we can solve for D,

$$0 = \frac{g}{c}(\frac{1}{c} + D)$$

consequently,

$$D = -\frac{g}{c^2}.$$

Substituting for D, we arrive at

$$s(t) = \frac{g}{c}(t + \frac{1}{c}e^{-ct} - \frac{1}{c})$$

So at $t = 10$ seconds, the object has fallen

$$s(10) = \frac{g}{c}(10 + \frac{1}{c}e^{-10c} - \frac{1}{c})$$

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