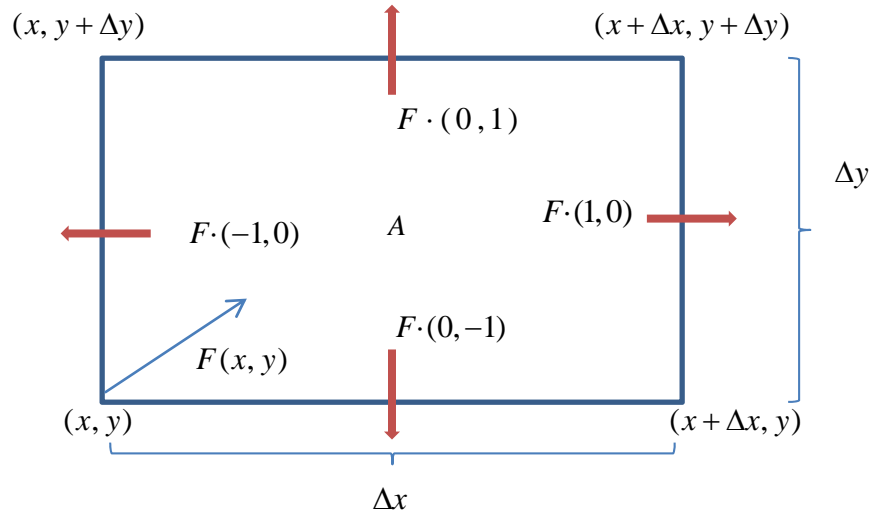


Divergence:

Let $F(x, y) = (f_1(x, y), f_2(x, y))$ be the velocity field of a fluid flowing in the plane, and that the first partial derivatives of f_1 and f_2 are continuous at each point in a region R . Suppose A is a small rectangle with one corner at (x, y) , whose sides are parallel to the coordinate axes, and such that $A \subset R$. Also, suppose that the side lengths are Δx and Δy .



Notice that the vector field is dotted with the vector in the direction of the red arrows to denote the rate at which the fluid is leaving the rectangle in that direction. To approximate the flow rate at the point (x, y) , we calculate the approximate flow rates across each edge in the directions of the red arrows, add these rates up, then divide the sum by the area of A . To get the actual area, we allow $\Delta x \rightarrow 0$ and $\Delta y \rightarrow 0$.

The fluid flow rate across the

Top	$v(x, y + \Delta y)\Delta x$
Bottom	$-v(x, y)\Delta x$
Right	$u(x + \Delta x, y)\Delta y$
Left	$-u(x, y)\Delta y$

When you add the top and bottom flow rates, you get

$$[v(x, y + \Delta y) - v(x, y)]\Delta x = \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y} \Delta y \Delta x$$

$$\approx \frac{\partial v}{\partial y} \Delta y \Delta x$$

When you add the right and left flow rates, you get

$$u(x + \Delta x, y) - u(x, y)\Delta y \approx \frac{\partial u}{\partial x} \Delta x \Delta y$$

The approximations are due to the continuity of the partial derivatives of u and v . Therefore, the flux across the boundary is given by $(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})\Delta x \Delta y$. Since we still have to divide by the area, we get

$$\text{div } F = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

DEFINITION The **divergence (flux density)** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \quad (1)$$

The figures below represent the velocity field of a gas flowing in the xy -plane. What is the divergence of each vector field and what is its physical meaning?

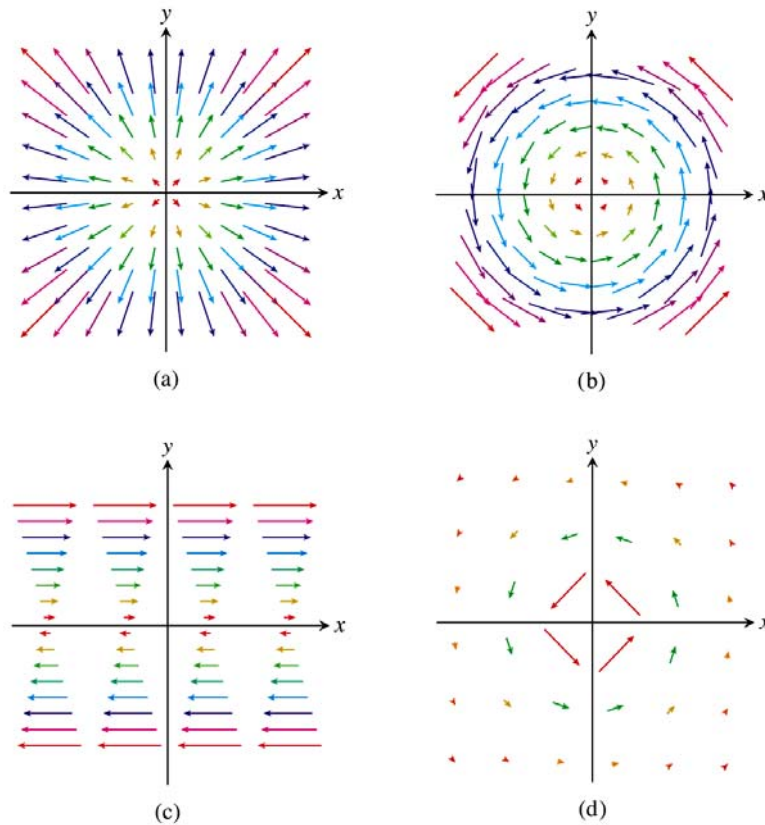


FIGURE 16.28 Velocity fields of a gas flowing in the plane (Example 1).

(a) Uniform expansion or compression: $F(x, y) = (cx, cy)$

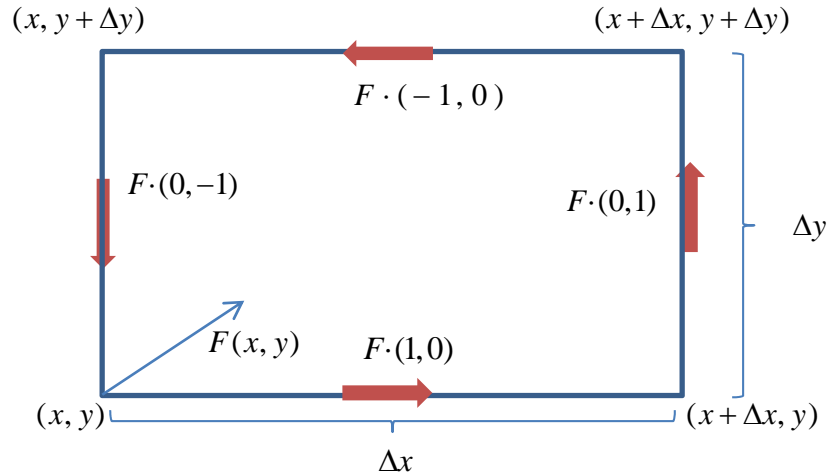
(b) Uniform rotation: $F(x, y) = (-cy, cx)$

(c) Shearing Flow: $F(x, y) = (y, 0)$

(d) Whirlpool Effect: $F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$

k-th component of Curl and Circulation Density

" This idea gives some sense of how the fluid is circulating around axes located at different points and perpendicular to the region." (Thomas, p. 934) This is called the circulation density, and we will later find out that it is the third coordinate function of a more general concept we call, the *curl*.



We use the exact same concepts as we did to compute the divergence.

Top	$-u(x, y + \Delta y)\Delta x$
Bottom	$u(x, y)\Delta x$
Right	$v(x + \Delta x, y)\Delta y$
Left	$-v(x, y)\Delta y$

When you add the top and bottom flow rates, you get

$$[u(x, y + \Delta y) + u(x, y)]\Delta x \approx -\frac{\partial u}{\partial y} \Delta y \Delta x$$

When you add the right and left flow rates, you get

$$[v(x + \Delta x, y) - v(x, y)]\Delta y \approx \frac{\partial v}{\partial x} \Delta x \Delta y$$

The approximations are due to the continuity of the partial derivatives of u and v . Therefore, the flux across the boundary is given by $(-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\Delta x \Delta y$. Since we still have to divide by the area, we get

$$\text{curl } \mathbf{F} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

DEFINITION The **circulation density** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is the scalar expression

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}. \quad (2)$$

This expression is also called **the k-component of the curl**, denoted by $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$.

The figures below represent the velocity field of a gas flowing in the xy -plane. What is the curl of each vector field and what is its physical meaning?

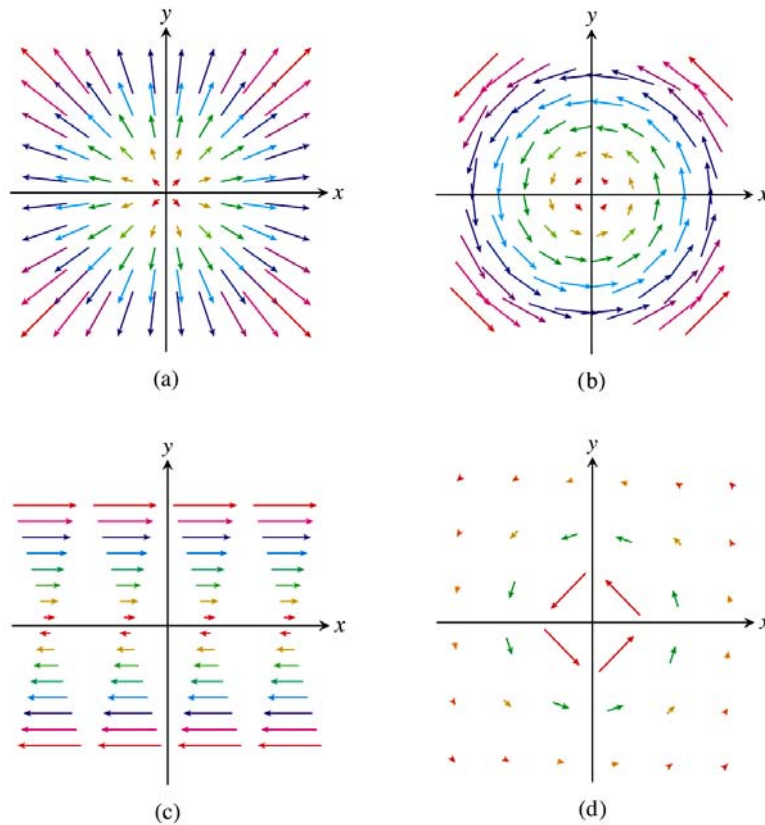


FIGURE 16.28 Velocity fields of a gas flowing in the plane (Example 1).

(a) Uniform expansion or compression: $F(x, y) = (cx, cy)$

(b) Uniform rotation: $F(x, y) = (-cy, cx)$

(c) Shearing Flow: $F(x, y) = (y, 0)$

(d) Whirlpool Effect: $F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$

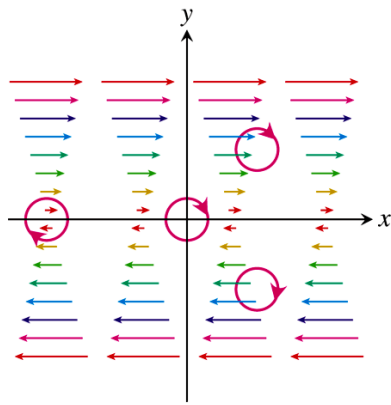
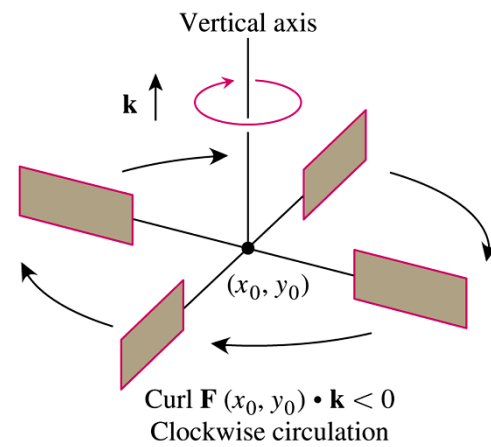
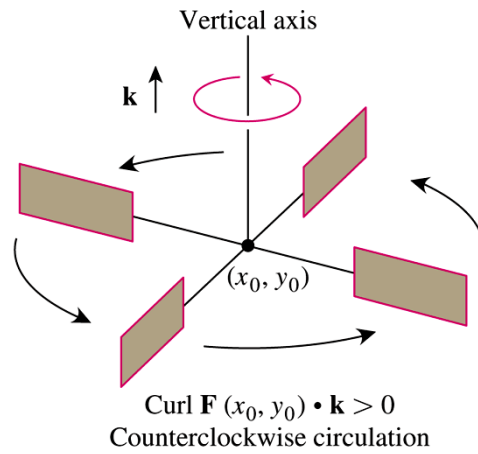


FIGURE 16.31 A shearing flow pushes the fluid clockwise around each point (Example 2c).

FIGURE 16.30 In the flow of an incompressible fluid over a plane region, the \mathbf{k} -component of the curl measures the rate of the fluid's rotation at a point. The \mathbf{k} -component of the curl is positive at points where the rotation is counterclockwise and negative where the rotation is clockwise.



"The k -th component of the curl at a point, P , gives a way to measure how fast and in what direction a small paddle wheel spins if it is put into the water at the point P with its axis perpendicular to the plane and parallel to $(0,0,1)$." (Thomas, p. 935)

Green's Theorem

Recall that in section 16.2, we defined the flow of a vector field, F , along a curve, C , and the outward pointing flux of a vector field, F , across a curve, C , with normal vector n is

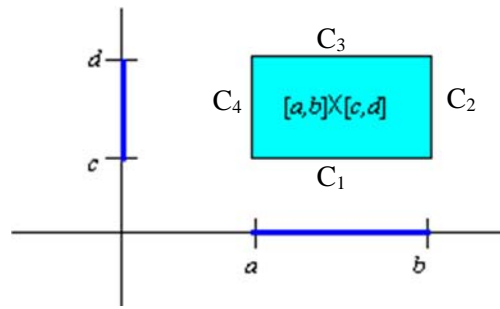
$$\text{Flux across } C \text{ of } F = \oint_C F \cdot n \, ds$$

Also, recall that we computed a differential form of the flux of F across a simple closed curve, C , given by

$$\text{Flux across } C \text{ of } F = \oint_C u \cdot dy - v \cdot dx$$

Green's Theorem relates the flux of a vector field across a simple closed curve with the divergence of the vector field, and the flow of a vector field along a simple closed curve with the circulation density of the vector field.

To see the relationship, let $R = \{(x, y) \mid x \in [a, b], y \in [c, d]\}$ be the rectangle with boundary paths, C_i . Let C be the concatenation of all the curves, C_i , which yields a counter-clockwise boundary path for R .



and $F(x, y) = (u(x, y), v(x, y))$. Then the integral $\iint_R \frac{\partial u}{\partial x} \, dx \, dy = \int_c^d \int_a^b \frac{\partial u}{\partial x} \, dx \, dy$ can be easily computed.

$$\begin{aligned} \int_c^d \int_a^b \frac{\partial u}{\partial x} \, dx \, dy &= \int_c^d u(x, y) \Big|_a^b \, dy \\ &= \int_c^d u(b, y) - u(a, y) \, dy \\ &= \int_c^d u(b, y) \, dy + \int_d^c u(a, y) \, dy \\ &= \int_{C_2} u \, dy + \int_{C_4} u \, dy \end{aligned}$$

Note that y is constant over the curves C_3 and C_1 , then

$$\int_{C_3} u \, dy = \int_d^c u \, dy = 0 = \int_c^d u \, dy = \int_{C_1} u \, dy .$$

Therefore,

$$\iint_R \frac{\partial u}{\partial x} \, dx \, dy = \int_{C_1} u \, dy + \int_{C_2} u \, dy + \int_{C_3} u \, dy + \int_{C_4} u \, dy = \oint_C u \, dy$$

A similar method can be used to show that

$$\iint_R \frac{\partial v}{\partial y} \, dx \, dy = -\oint_C v \, dx$$

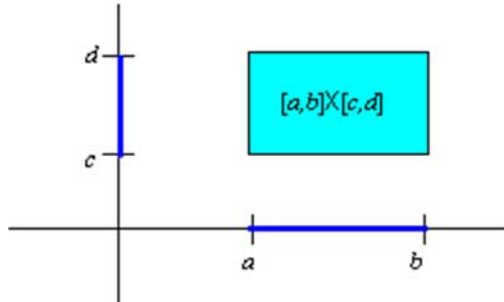
Greens Theorem for Flux:

The double integral of the divergence of a vector field across the boundary of a simply connected domain is the line integral of the Flux along the boundary.

$$\int \int_R \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} dx dy = \int_{\partial R} u dy - v dx$$

Greens Theorem for Circulation:

The double integral of the k-th component of the curl inside a simply connected region is the line integral of counterclockwise circulation of F along the boundary of the simply connected region is the



$$\int \int_R \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} dx dy = \int_{\partial R} u dx + v dy$$

The proof this theorem is left as an exercise to the student. Yes, that means its homework. Due Monday.

The connection between real and complex line integrals.

Suppose $f(z) = u(x, y) + iv(x, y)$

Then the complex line integral can be expressed as real line integrals because

$$\begin{aligned} \int_C f(z) dz &= \int_C (u + iv)(dx + idy) \\ &= \int_C u dx + iv dx + iu dy - v dy \\ &= \int_C u dx - v dy + i \int_C v dx + u dy \end{aligned}$$

By both Green's Theorems, this is equivalent to

$$\begin{aligned} \int_C f(z) dz &= \int_C u dx - v dy + i \int_C v dx + u dy \\ &= \iint_R \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} dx dy + i \iint_R \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} dx dy \end{aligned}$$

The Cauchy Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Which means all analytic functions are conservative vector fields!