Example 1. Let $\phi : \mathbb{Z}_{15} \to \mathbb{Z}_5$ defined by $\phi(n) = n \mod (5)$.
Then $\phi^{-1}(0) = \{0, 5, 10\}$ because

$$\phi(0) = 0 \mod (5)$$
$$\phi(5) = 0 \mod (5)$$
$$\phi(10) = 0 \mod (5)$$

We call the preimage of 0, $\ker(\phi)$.
Therefore, $\ker(\phi) =< 5 >$ is the group generated by the element 5 in $\mathbb{Z}_{15}$.

What is the preimage of $1 \mod (5)$?
Since

$$\phi(1) = 1 \mod (5)$$
$$\phi(6) = 1 \mod (5)$$
$$\phi(11) = 1 \mod (5)$$

Then $\phi^{-1} = \{1, 6, 11\} = 1 + \{0, 5, 10\} = 1+ < 5 >= 1 + \ker(\phi)$.

Similarly,

$$\phi^{-1}(2) = 2 + \ker\phi$$
$$\phi^{-1}(3) = 3 + \ker\phi$$
$$\phi^{-1}(4) = 4 + \ker\phi$$

We call these sets the cosets of the kernel.
A left coset is a set

$$g + \ker\phi = \{g + k \mid k \in \ker\phi\}$$

and a right coset is a set

$$\ker\phi + g = \{k + g \mid k \in \ker\phi\}$$

If the group is abelian then the right cosets always equal the left cosets.

Example 2. Let $f : \mathbb{R} \to \mathbb{C}*$ be defined by

$$f(x) = \cos(x) + i\sin(x)$$

Note that because I used the group of all real numbers for the domain, it is implied that the operation is addition. Therefore, the identity$\mathbb{R}$ is 0.
However, the fact that $\mathbb{C}*$ is the codomain suggests that the operation is multiplication is 1, and corresponds to the point $(1, 0)$. Also, notice that the image of $\mathbb{R}$ is the unit circle in $\mathbb{C}$, and the homomorphism "wraps" the real numbers around the unit circle infinitely many times. What is ker$f$?
By definition it is the preimage of 1.

$$\begin{aligned} \ker f &= f^{-1}(1) \\ &= \{0, 2\pi, 4\pi, \dots\} \\ &= \{2k\pi \mid k \in \mathbb{Z}\} \\ &=< 2\pi > \end{aligned}$$

Similarly, if I wanted to look at the preimage of any point on the unit circle, $a + ib$, then I would need to find the solution, x, in $[0, 2\pi)$ to the system of equations

$$cos(x) = a$$

$$sin(x) = b$$

and the preimage $f^{-1}(\{a + ib\}) = \{x + 2k\pi \mid k \in \mathbb{Z}\} = x+ < 2\pi >$.
Note this homomorphism is surjective.

For the following function, assume it's a homomorphism. Why does the assignment below completely determine the homomorphism. In other words, how do you know the image of $(0,0)$ and $(1,1)$.?

Example 3. Let $f : \mathbb{Z}_2 \times \mathbb{Z}_2 \to S_4$ be defined by

$$f((0,1)) = (12)(34)$$

and

$$f((1,0)) = (13)(24)$$

Can this be a homomorphism?
What is the image of $Z_2 \times Z_2$?

What is $ker(f)$?

Is this an injective homomorphism?

Example 4. Let $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous } \}$ be the set of all continuous functions. Then $\mathcal{F}$ is a group under addition. (How about multiplication or composition?)

Show that $\dfrac{d}{dx} : \mathcal{F} \to \mathcal{F}$ is a homomorphism.

What is the kernel?

Is it surjective of injective?

What are the cosets of the kernel?