- Implicit Differentiation:
  - Consider y as a function of x
  - Whenever you take a derivative of a y, multiply by dy/dx.
  - If y is next to x, make sure to use product rule.
- Finding higher derivatives ($\frac{d^2y}{dx^2}$)
  - Implicitly differentiate again.
  - The derivative of dy/dx is $\frac{d^2y}{dx^2}$
  - Plug in dy/dx in terms of y and x
- Meaning
  - $\frac{dy}{dx}$ is the slope of the curve given by the equation at a point (a, b)
  - $\frac{d^2y}{dx^2}$ is the concavity of the curve given by the equation at a point (a, b)
- Tangent Lines
  - Horizontal: $\frac{dy}{dx} = 0$
  - Vertical: $\frac{dy}{dx}$ does not exist (denominator = 0)
  - Critical Point: $\frac{dy}{dx} = 0$ or does not exist
  - Over-approximation: $\frac{d^2y}{dx^2} < 0$ because curve is concave down
  - Under-approximation: $\frac{d^2y}{dx^2} > 0$ because curve is concave up

Also, since $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

2000 (Non-Calc)

Consider the curve given by $xy^2 - x^3y = 6$.

(a) Show that $\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}$.

(b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

(a) $y^2 + 2xy\dfrac{dy}{dx} - 3x^2y - x^3\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}(2xy - x^3) = 3x^2y - y^2$

$\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$

2 $\begin{cases} 1: \text{implicit differentiation} \\ 1: \text{verifies expression for } \dfrac{dy}{dx} \end{cases}$

(b) When $x = 1$, $y^2 - y = 6$

$y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3,\ y = -2$

At $(1,3)$, $\dfrac{dy}{dx} = \dfrac{9-9}{6-1} = 0$

Tangent line equation is $y = 3$

At $(1,-2)$, $\dfrac{dy}{dx} = \dfrac{-6-4}{-4-1} = \dfrac{-10}{-5} = 2$

Tangent line equation is $y + 2 = 2(x-1)$

4 $\begin{cases} 1: y^2 - y = 6 \\ 1: \text{solves for } y \\ 2: \text{tangent lines} \end{cases}$

Note: 0/4 if not solving an equation of the form $y^2 - y = k$

(c) Tangent line is vertical when $2xy - x^3 = 0$

$x(2y - x^2) = 0$ gives $x = 0$ or $y = \dfrac{1}{2}x^2$

There is no point on the curve with x-coordinate 0.

When $y = \dfrac{1}{2}x^2$, $\dfrac{1}{4}x^5 - \dfrac{1}{2}x^5 = 6$

$-\dfrac{1}{4}x^5 = 6$

$x = \sqrt[5]{-24}$

3 $\begin{cases} 1: \text{sets denominator of } \dfrac{dy}{dx} \text{ equal to } 0 \\ 1: \text{substitutes } y = \dfrac{1}{2}x^2 \text{ or } x = \pm\sqrt{2y} \\ \quad \text{into the equation for the curve} \\ 1: \text{solves for } x\text{-coordinate} \end{cases}$

2004 (Non-Calc)

4. Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

   (a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

   (b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

   (c) Find the value of $\frac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

(a) $2x + 8yy' = 3y + 3xy'$ $(8y - 3x)y' = 3y - 2x$ $y' = \dfrac{3y - 2x}{8y - 3x}$	$2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{solves for } y' \end{cases}$
(b) $\dfrac{3y - 2x}{8y - 3x} = 0;\ 3y - 2x = 0$ When $x = 3,\ 3y = 6$ $y = 2$ $3^2 + 4 \cdot 2^2 = 25$ and $7 + 3 \cdot 3 \cdot 2 = 25$ Therefore, $P = (3, 2)$ is on the curve and the slope is 0 at this point.	$3 : \begin{cases} 1 : \dfrac{dy}{dx} = 0 \\ 1 : \text{shows slope is 0 at } (3, 2) \\ 1 : \text{shows } (3, 2) \text{ lies on curve} \end{cases}$
(c) $\dfrac{d^2y}{dx^2} = \dfrac{(8y - 3x)(3y' - 2) - (3y - 2x)(8y' - 3)}{(8y - 3x)^2}$ At $P = (3, 2)$, $\dfrac{d^2y}{dx^2} = \dfrac{(16 - 9)(-2)}{(16 - 9)^2} = -\dfrac{2}{7}$. Since $y' = 0$ and $y'' < 0$ at P, the curve has a local maximum at P.	$4 : \begin{cases} 2 : \dfrac{d^2y}{dx^2} \\ 1 : \text{value of } \dfrac{d^2y}{dx^2} \text{ at } (3, 2) \\ 1 : \text{conclusion with justification} \end{cases}$

2005B (Non-Calc)

Consider the curve given by $y^2 = 2 + xy$.

(a) Show that $\frac{dy}{dx} = \frac{y}{2y - x}$.

(b) Find all points (x, y) on the curve where the line tangent to the curve has slope $\frac{1}{2}$.

(c) Show that there are no points (x, y) on the curve where the line tangent to the curve is horizontal.

(d) Let x and y be functions of time t that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of y is 3 and $\frac{dy}{dt} = 6$. Find the value of $\frac{dx}{dt}$ at time $t = 5$.

(a) $2yy' = y + xy'$

$(2y - x)y' = y$

$y' = \dfrac{y}{2y - x}$

$2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{solves for } y' \end{cases}$

(b) $\dfrac{y}{2y - x} = \dfrac{1}{2}$

$2y = 2y - x$

$x = 0$

$y = \pm\sqrt{2}$

$(0, \sqrt{2}), (0, -\sqrt{2})$

$2 : \begin{cases} 1 : \dfrac{y}{2y - x} = \dfrac{1}{2} \\ 1 : \text{answer} \end{cases}$

(c) $\dfrac{y}{2y - x} = 0$

$y = 0$

The curve has no horizontal tangent since $0^2 \neq 2 + x \cdot 0$ for any x.

$2 : \begin{cases} 1 : y = 0 \\ 1 : \text{explanation} \end{cases}$

(d) When $y = 3$, $3^2 = 2 + 3x$ so $x = \dfrac{7}{3}$.

$\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = \dfrac{y}{2y - x} \cdot \dfrac{dx}{dt}$

At $t = 5$, $6 = \dfrac{3}{6 - \frac{7}{3}} \cdot \dfrac{dx}{dt} = \dfrac{9}{11} \cdot \dfrac{dx}{dt}$

$\left.\dfrac{dx}{dt}\right|_{t=5} = \dfrac{22}{3}$

$3 : \begin{cases} 1 : \text{solves for } x \\ 1 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$