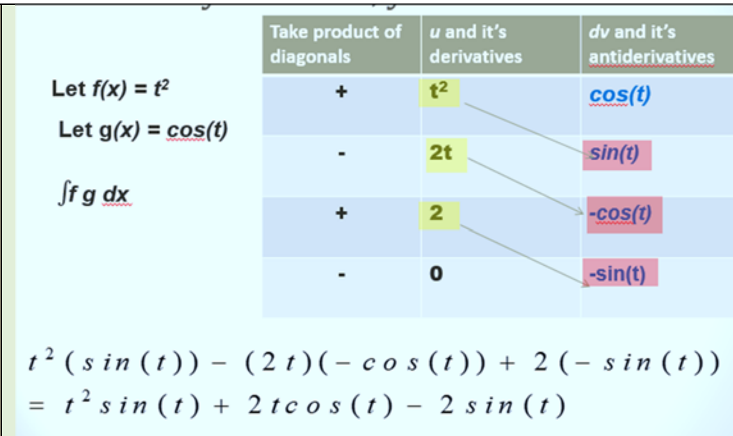
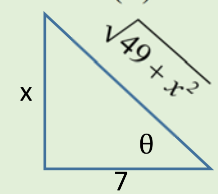
Integration Technique Cheat Sheet

Integration Technique	***When to use this method***	***The Method***	***Example***
U-sub	When you have reverse chain rule $\int f(u(x))u'(x)dx$	Choose u and compute du so that $\int f(u)du$ is integrable.	$\int \frac{x^2}{\sqrt{4x^3+2}}dx \quad u=4x^3+2 \quad du=12x^2dx$ $=\frac{1}{12}\int \frac{12x^2}{\sqrt{4x^3+2}}dx=\frac{1}{12}\int\frac{du}{\sqrt{u}}$
Trigonometric Identities	When you are given $\int \sin^k(x)dx$ or $\int \cos^k(x)dx$	k is even = use the half angle formulas $\cos^2(x)=\frac{1+\cos(2\theta)}{2}$ $\sin^2(x)=\frac{1-\cos(2\theta)}{2}$ k is odd = "save" one of the functions for the du, and substitute the rest with the identities $\cos^2(x)=1-\sin^2(x)$ $\sin^2(x)=1-\cos^2(x)$	$\int \cos^4(x)dx=\int(\frac{1+\cos(2\theta)}{2})^2d\theta$ $=\int\frac{1}{4}(1+2\cos(2\theta)+\cos^2(2\theta))d\theta$ $=\int\frac{1}{4}(1+2\cos(2\theta)+\frac{1+\cos(4\theta)}{2})d\theta$ or $\int\cos^3(x)dx=\int(1-\sin^2(x))\cos(x)dx$ $=\int 1-u^2du$
Integration by Parts (traditional)	When you have the product of two functions but neither is the derivative of the other $\int f(x)g(x)dx$	Choose $u=f(x)$ and $dv=g(x)$ $du=f'(x) \quad v=\int g(x)dx$ $\int udv=uv-\int vdu$	Choosing u (log, inverse trig, algebraic, trig, exp) $\int\sin(x)e^xdx=\sin(x)e^x-\int e^x\cos(x)dx$ $u_1=\sin(x) \quad dv_1=e^x \qquad u_2=\cos(x) \quad dv_2=e^x$ $du_1=\cos(x) \quad v_1=e^x \qquad du_2=-\sin(x) \quad v_2=e^x$ $\int\sin(x)e^xdx=\sin(x)e^x-\cos(x)e^x-\int\sin(x)e^xdx$ $2\int\sin(x)e^xdx=\sin(x)e^x-\cos(x)e^x$ $\int\sin(x)e^xdx=\frac{1}{2}e^x(\sin(x)-\cos(x))$

Integration by Parts (tabular)	When one function is a polynomial and the other is NOT a log function	Sign | u (der) | dv (int) + | u | dv - | u' | v + | u'' | $\int v$	
Partial Fractions	When you are integrating a rational function in which the numerator is not the derivative of the denominator	You are given A and B and factor the denominator into $\frac{Ax+B}{(x-r_1)(x-r_2)} = \frac{M}{x-r_1} + \frac{N}{x-r_2}$ $Ax+B = M(x-r_2)+N(x-r_1)$ Plug in $x=r_1$ and $x=r_2$	$\int \frac{4x+37}{x^2+11x+28}dx$ $\frac{4x+37}{x^2+11x+28} = \frac{A}{x+7} + \frac{B}{x+4}$ $4x+37 = A(x+4)+B(x+7)$ $x=-4 \to B=7 \quad x=-7 \to A=-3$ $\int \frac{4x+37}{x^2+11x+28}dx = \int -\frac{3}{x+7}dx + \int \frac{7}{x+4}dx$ $= -3\ln\|x+7\| + 7\ln\|x+4\| + C$
Trigonometric Substitution	When you have $\sqrt{a-x^2}$ or $\sqrt{x^2-a}$ or $\sqrt{x^2+a}$	Using the trig identities $1-\sin^2(\theta) = \cos^2(\theta)$ $1+\tan^2(\theta) = \sec^2(\theta)$ $\sec^2(\theta) - 1 = \tan^2(\theta)$ Choose x to be one of the trig functions so that the square root is cancelled.	$\int\sqrt{49+x^2}dx \quad x = 7\tan(\theta) \quad dx = 7\sec^2(\theta)$ $\int\sqrt{49+49\tan^2(\theta)}7\sec^2(\theta)d\theta$ $\int 47\sec^3(x)dx$ 

Integration Technique Cheat Sheet

Tricky Integrals:

1. $\int \sec(x)dx$

$$\int \sec(x)dx = \int \sec(x)\cdot\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}dx$$

$$= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}dx$$

$$u = \sec(x)+\tan(x) \qquad du = \sec(x)\tan(x)+\sec^2(x)$$

$$= \int \frac{du}{u} = \ln|\sec(x)+\tan(x)|+C$$

2. $\int \frac{x}{x+1}dx$

$$\int \frac{x}{x+1}dx = \int \frac{x+1-1}{x+1}dx$$

$$= \int \frac{x+1}{x+1} - \frac{1}{x+1}dx$$

$$= \int 1 - \frac{1}{x+1}dx$$

$$= x - \ln|x+1|+C$$

3. $\int \sec^3(x)dx$

$$\int \sec^3(x)dx = \int \sec^2(x)\sec(x)dx$$

Integration by Parts

$$u = \sec(x) \qquad dv = \sec^2(x)dx$$

$$du = \sec(x)\tan(x) \qquad v = \tan(x)dx$$

$$\int \sec^3(x)dx = \sec(x)\tan(x) - \int \tan^2(x)\sec(x)dx$$

$$\int \sec^3(x)dx = \sec(x)\tan(x) - \int (\sec^2(x)-1)\sec(x)dx$$

$$\int \sec^3(x)dx = \sec(x)\tan(x) - \int \sec^3(x)dx + \int \sec(x)dx$$

$$\int \sec^3(x)dx = \sec(x)\tan(x) - \int \sec^3(x)dx + \ln|\sec(x)+\tan(x)|+C$$

$$2\int \sec^3(x)dx = \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|+C$$

$$\int \sec^3(x)dx = \frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|)+C$$

4. $\int \ln(x)dx$

$$\int \ln(x)dx$$

Integration by Parts

$$u = \ln(x) \qquad dv = dx$$

$$du = \frac{1}{x}dx \qquad v = x$$

$$\int \ln(x)dx = x\ln(x) - \int dx$$

$$= x\ln(x) - x + C$$

Trig Substitution

L. Marizza A. Bailey

Example 1.

$$\int \frac{1}{x\sqrt{4+x^2}}dx$$

Let $x = 2\tan(\theta)$ then $dx = 2\sec^2(\theta)d\theta$.

$$\int \frac{1}{2\tan(\theta)\sqrt{4+4\tan^2(\theta)}}\sec^2(\theta)d\theta$$

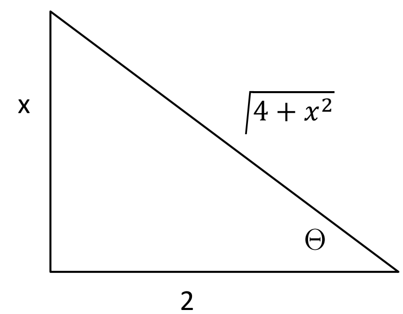
$$= \int \frac{\sec^2(\theta)}{2\tan(\theta)2\sec(\theta)}d\theta$$

$$= \frac{1}{4}\int \sec(\theta)\cot(\theta)d\theta$$

$$= \frac{1}{4}\int \frac{1}{\cos(\theta)}\frac{\cos(\theta)}{\sin(\theta)}d\theta$$

$$= \frac{1}{4}\int \csc(\theta)d\theta$$

$$= -\frac{1}{4}\ln(\csc(\theta)+\cot(\theta)) + C$$



Since x =2 tan(Θ), then tanΘ = $\frac{x}{2}$.
Therefore

$$\csc(\theta) = \frac{\sqrt{4+x^2}}{x} \text{ and } \cot(\theta) = \frac{2}{x}$$

Hence, the integral is

$$= -\frac{1}{4}\ln(\csc(\theta)+\cot(\theta)) + C$$

$$= \frac{-1}{4}\ln(\frac{\sqrt{4+x^2}+2}{x}) + C$$

$$= \frac{-1}{4}\ln(\sqrt{4+x^2}+2) + \frac{1}{4}\ln(x) + C$$

Notes:

- If you see $\sqrt{a+x^2}$, you will need to use $x = \sqrt{a}\tan(\theta)$
- If you see $\sqrt{a-x^2}$, you will need to use $x = \sqrt{a}\sin(\theta)$
- If you see $\sqrt{x^2-a}$, you will need to use $x = \sqrt{a}\sec(\theta)$
- Remember the magic box: $x = \tan(y)$ implies $y = \arctan(x)$
- After you finish integrating with respect to theta, take your trig substitution and solve for the trig function. This will help you draw your triangle.

SOME MORE TECHNIQUES OF INTEGRATION

L. MARIZZA A. BAILEY

1. Partial Fractions

Suppose your model requires for population with limited resources:

The rate of increase of the population at time t is directly proportional to both P and L-P, where L is the maximum size of the population

This is called the *logistic growth model.* This is modeled by the equation

$$\frac{dP}{dt} = kP(L-P).$$

To find the general solution to the differential equation, we must separate the variables:

$$\frac{dP}{P(L-P)} = kdt$$

Now we need to integrate. If we use u-substitution on the left side, we would use

$$u = P(L-P) = PL - P^2,$$

then $du = L - 2P$. This, however, does not seem to help at all. This is why we need to split up the fraction . It would be nice if there existed $A, B \in \mathbb{R}$ such that

$$\frac{1}{P(L-P)} = \frac{A}{P} + \frac{B}{L-P}$$

Before we tackle the problem above, let us look at simpler rational functions. Consider $f(x) = \dfrac{1}{(x-1)(2x-3)}$.
How do we find,

$$f(x) = \int \frac{1}{(x-1)(2x-3)}?$$

The same problem arises. We need to find A, B such that

$$\frac{1}{(x-1)(2x-3)} = \frac{A}{x-1} + \frac{B}{2x-3}$$

By multiplying both sides by $(x-1)(2x-3)$ we get

$$1 = A(2x-3) + B(x-1)$$

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If this A and B exist, then they make the equation true regardless of x. In particular, this would be true for $x = 1$,

$$\begin{aligned} 1 &= A(2(1) - 3) + B(1 - 1) \\ 1 &= -A \end{aligned}$$

Now we know that $A = -1$.
Similarly by allowing $x = 0$, we get

$$\begin{aligned} 1 &= A(2(0) - 3) + B(0 - 1) \\ 1 &= (-1)(-3) + B(-1) \\ 1 &= 3 - B \\ -2 &= -B \\ 2 &= B \end{aligned}$$

Therefore, we have

$$\frac{1}{(x-1)(2x-3)} = \frac{-1}{x-1} + \frac{2}{2x-3}$$

To check your answer, plug our A and B into the equation

$$\begin{aligned} 1 &= A(2x - 3) + B(x - 1) \\ &= (-1)(2x - 3) + 2(x - 1) \\ &= -2x + 3 + 2x - 2 \\ &= 1 \end{aligned}$$

Now that we have split up the fraction we can integrate:

$$\begin{aligned} \int \frac{1}{(x-1)(2x-3)} dx &= \int \frac{-1}{x-1} dx + \frac{2}{2x-3} dx \\ &= -\ln(x-1) + \ln(2x-3) \end{aligned}$$

This technique is called *partial fractions*.
Try to solve the following integral using partial fractions:

Problem 1.

$$\int \frac{4}{(x-1)(x-4)} dx$$

Now let us see how to solve the logistic growth model.

$$\frac{1}{P(L-P)} = \frac{A}{P} + \frac{B}{L-P}$$
$$1 = A(L-P) + BP$$

Allowing $P = L$ (remember L is just a number), and $P = 0$ would make our computations much easier.
If $P = L$,

$$1 = BL$$

or

$$\frac{1}{L} = B$$

If $P = 0$,

$$1 = A(L)$$

or

$$\frac{1}{L} = A$$

Hence,

$$\frac{1}{P(L-P)} = (\frac{1}{L})(\frac{1}{P} + \frac{1}{L-P})$$

Now to solve for P

$$\int \frac{1}{P(L-P)} dP = \int kdt$$
$$\frac{1}{L} \int \frac{1}{P} dP + \int \frac{1}{L-P} dP = \int kdt$$
$$\frac{1}{L} \ln(P) - \ln(L-P) = kt + C$$

Now we begin the grueling process of solving for P. Remember L is the limiting factor, and k and C are constants.

$$\ln(\frac{P}{L-P}) = Lkt + C$$
$$\frac{P}{L-P} = Ce^{Lkt}$$
$$\frac{L-P}{P} = \frac{C}{e^{Lkt}}$$

In the last step, we took the reciprocal of both sides in order to be able to break the fraction up into more pieces.

$$\begin{aligned}\frac{L}{P} - 1 &= \frac{C}{e^{Lkt}} \\ \frac{L}{P} &= \frac{C}{e^{Lkt}} + 1 \\ \frac{L}{P} &= \frac{C + e^{Lkt}}{e^{kLt}} \\ \frac{P}{L} &= \frac{e^{kLt}}{C + e^{kLt}} \\ P &= \frac{Le^{kLt}}{C + e^{kLt}}\end{aligned}$$

Now, we factor out e^{kLt} from the numerator and denominator of the right hand side,

$$P = \frac{L}{Ce^{-kLt} + 1}$$

Given two initial conditions, we can solve for k and C.

Let us attempt to apply this model to the following problem:

Problem 2. In a town of 100,000, there were 20000 residents which heard of a radio announcement about the local political scandal. The rate of growth of the spread of information about the scandal is jointly proportional to the number of people who heard it and the number of people who had not heard it. If 50% of the population heard about the scandal after one hour, how long was it until 80% of the population heard it?

(a) Step 1: Write the differential equation.

(b) Step 2: Separate Variables.

(c) Step 3: Use Partial Fractions to write in an integrable form.

(d) Step 4: Integrate Both Sides. [Don't forget the C]

(e) Step 5: Solve for P.

(f) Step 6: Plug in your initial conditions.

2. Integration by Parts

The velocity of a particle is given by $v(t) = (\cos(t), te^t)$. What is the position function of the particle with respect to time if the initial position of the particle was $(1, 2)$?

In order to solve this problem, we need to find the anti-derivative of both the y and x velocity functions.

$$\begin{aligned} x(t) &= \int \cos(t)dt \\ &= \sin(t) + C \end{aligned}$$

Since at $t = 0$ the x coordinate was 1, we can substitute these values:

$$1 = \sin(0) + C$$

implies $C = 1$. Therefore, the x-coordinate function is given by $x(t) = \sin(t)+C$.

Now, let us solve for the y-position function.

$$y(t) = \int te^t dt$$

Can we use u-substitution? Partial Fractions? This looks like the product of two functions. Can we use product rule? Let us review product rule, and see if it gives us any ideas.

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + \frac{dv}{dx}u$$

Integrating both sides, we achieve

$$uv = \int vdu + \int udv$$

Solving for one of the integrals,

$$\int udv = uv - \int vdu$$

This formula is called *Integration by Parts* .
How can we use this to help us solve the integral above?

Let $u = t$ and $dv = e^t$.
Then $du = dt$ and $v = e^t$.
Then by the formula,

$$\begin{aligned} \int udv &= \int te^t dt \\ &= te^t - \int e^t dt \\ &= te^t - e^t + C \end{aligned}$$

By plugging in are initial condition, $t = 0$, then $y = 2$, we get

$$2 = y(0) = 0e^0 - e^0 + C$$

Simplification yields

$$2 = C - 1$$

Thus $C = 3$ and $y(t) = te^t - e^t + 3$. The position function is given by

$$(x(t), y(t)) = (\sin(t) + 1, te^t - e^t + 3)$$

Department of Mathematics, Arkansas School of Mathematics, Sciences and the Arts
E-mail address: `baileym@asmsa.com`

TRIG TECHNIQUES

L. MARIZZA A. BAILEY

1. Trig Identities

Here are some examples that you may encounter while solving integrals with trigonometric functions.

Problem 1.

$$\int \sin^3(x)\cos^2(x)dx$$

Solution 1. If $\sin^3(x)$ were a $\sin(x)$, we could use u-substitution. Alas, it is not. So, we will substitute $1-\cos^2(x)$ for a $\sin^2(x)$ and leave the $\sin(x)$ as an odd man out. We will be left with a polynomial in $\cos(x)$, and a $\sin(x)dx$ term perfectly positioned to be a du for $u = \cos(x)$.

$$\begin{aligned}
\int \sin^3(x)\cos^2(x)dx &= \int \sin(x)(1-\cos^2(x))\cos^2(x)dx \\
&= \int \sin(x)(\cos^2(x)-\cos^4(x))dx \\
&= -\int u^2 - u^4 du \qquad \text{letting } u = \cos(x) \text{ and } du = -\sin(x)dx. \\
&= -\frac{u^3}{3} + \frac{u^5}{5} + C \\
&= -\frac{\cos^3(x)}{3} + \frac{\cos^5(x)}{5} + C
\end{aligned}$$

Problem 2.

$$\int \sin^2(x)\cos^2(x)dx$$

Solution 2. Unfortunately, writing $\sin^2(x) = 1-\cos^2(x)$ will only leave us with a mess of $\cos^2(x)$ to which we would have to individually decrease the exponent by using the half angle trig identity.

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)dx &= \int \frac{1-\cos(2x)}{2}\cdot\frac{1+\cos(2x)}{2}dx \\
&= \frac{1}{4}\int 1-\cos^2(2x)dx \\
&= \frac{1}{4}\int 1-\frac{1+\cos(4x)}{2}dx \\
&= \frac{1}{4}\int \frac{1}{2}-\frac{1}{2}\cos(4x)dx \\
&= \frac{1x}{8} - \frac{\sin(4x)}{32} + C
\end{aligned}$$

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Problem 3.

$$\int \tan(x)\sec^4 x dx$$

Solution 3. This integral would be much friendlier if there was a tangent nearby, so that $u = \tan(x)$ and $du = \sec^2(x)dx$. If I break the integrand into $\sec^2(x)(x)\sec^2(x)$ and substitute $\sec^2(x) = (1 + \tan^2(x))$, I will only have a $\sec^2(x)$ left. Let us proceed, and see what happens.

$$\begin{aligned}
\int \tan(x)\sec^4(x)dx &= \int \tan(x)(1+\tan^2(x))\sec^2(x)dx \\
&= \int (\tan(x) + \tan^3(x))\sec^2(x)dx \qquad \text{Let } u = \tan(x) \text{ then } du = \sec^2(x) \\
&= \int u + u^3 du \\
&= \frac{\tan^2(x)}{2} + \frac{\tan^4(x)}{4} + C
\end{aligned}$$

Problem 4.

$$\int \sec^3(x)dx$$

Solution 4. This integral is rather difficult in the sense that substituting a $\sec^2(x) = 1 + \tan^2(x)$ will not help because there is only one secant left, and it would not help to make a u-substitution for $u = \tan(x)$. So instead we will try integration by parts. Let $u = \sec(x)$ and $dv = \sec^(x)dx$.
Then $du = \sec(x)\tan(x)$ and $v = \tan(x)$.

$$\int \sec^3(x)dx = \sec(x)\tan(x) - \int \tan^2(x)\sec(x)dx$$

Let $u = \tan(x)$ and $dv = \tan(x)\sec(x)dx$. Then $du = \sec(x)\tan(x)$ and $v = \sec(x)$.

$$\int \sec^3(x) = \sec(x)\tan(x) - [\tan(x)\sec(x) - \int \sec^2(x)\tan(x)dx]$$

$$\int \sec^3(x) = \sec(x)\tan(x) - \tan(x)\sec(x) + \int \sec(x)\sec(x)\tan(x)dx$$

We can use u-substitution.
Let $u = \sec(x)$, then $du = \sec(x)\tan(x)dx$. Then

$$\int \sec^3(x) = \sec(x)\tan(x) - \tan(x)\sec(x) + \int u du$$

$$\int \sec^3(x) = \sec(x)\tan(x) - \tan(x)\sec(x) + \frac{\sec^2(x)}{2} + C$$

The bottom line is that you want to use your brain and creativity to solve these integrals, and most importantly, you need to know your trig identities.

Department of Mathematics, Arkansas School of Mathematics and Sciences.
E-mail address: baileym@asmsa.org

Techniques of Integration and Differential Equation Worksheet
Name___________________________________

Evaluate the integral.

1) $\int 2xe^{x}\, dx$

Integration by Parts:

2) $\int x^{7} \sec x^{8}\, dx$

U – Substitution:

3) $\int \frac{6 \csc^{3} x}{\tan x} dx$

Trigonometric Identities:

4) $\int_{0}^{1} \frac{dx}{\sqrt{64 - x^{2}}}$

Trigonometric Substitution:

5) $\int_{0}^{\ln 3} \frac{e^t\,dt}{16 + e^{2t}}$

U – substitution, then trigonometric substitution: Don't forget to change the limits!!!!

Integrate the function.

6) $\int_{-1}^{1} \frac{6}{1 + 36t^2}\,dt$

Trigonometric Substitution: Remember to make a referenece triangle and rewrite the integratl in terms of t before evaluating:

7) $\int_{0}^{\pi/2} \cos^2 3x \sin^3 3x\,dx$

Trigonometric Identities:

Express the integrand as a sum of partial fractions and evaluate the integral.

8) $\int \frac{x + 9}{x^2 + 5x}\, dx$

Partial Fractions:

9) $\int_8^9 \frac{12}{x^2 - 36}\, dx$

Partial Fractions:

10) $\int \frac{36\, dx}{x^3 - 9x}$

Partial Fractions

1) $2xe^x - 2e^x + C$

2) $\frac{1}{8}\ln\left|\sec x^8 + \tan x^8\right| + C$

3) $-2\csc^3 x + C$

4) $\sin^{-1}\frac{1}{8}$

5) 0.100

6) $2\tan^{-1} 6$

7) $\frac{2}{45}$

8) $\frac{1}{5}\ln\left|\frac{x^9}{(x+5)^4}\right| + C$

9) 0.336

10) $-4\ln|x| + 2\ln|x-3| + 2\ln|x+3| + C$