INTRODUCTION TO METRIC SPACES

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We already have a intuitive idea of what it means to be an open set. Like an open room, it has no boundary. It is not necessary to imagine a boundary in the physical world, because we can usually see it; a wall, a door, or a line drawn on the ground. It impedes us from progressing out of the area which is bounded. Sometimes boundaries may be abstract, and it is necessary to imagine them; national borders, county lines, property boundaries, are limits we have imposed on the physical world.

Now, let us consider physical objects with no boundary. If we are confronted with a fenceless prairie, or "open" space, or even a room without walls, we consider these to be "open" because we can "get out".The commonality being that we cannot reach the "boundary" regardless of how far out we reach. This leads us to the concept of distance.

Mathematically, we would like to abstract the necessary properties of this definition of open to mathematical models of the physical world, and then, if possible, to all worlds.

1. Metric Spaces

The word metric is familiar to us when measuring the length of objects. In mathematics, we use the word metric to describe a function which has the properties of the distance function we are familiar with.

Let us extract the properties of distance:

Definition 1 (metric)**.** Let X be a set.
A *metric*, d on a set X is a real-valued function on $X \times X$ such that

- **(a)** The distance between any two points is always positive, $d(x, y) \geq 0$.
- **(b)** The distance between a point and itself is zero $d(x, x) = 0$.
- **(c)** Distance is invariant under order $d(x, y) = d(y, x)$. [Distance is symmetric]
- **(d)** If you are traveling from a point x to a point z, but first travel to point y, you will either increase or not vary the distance traveled.

$$d(x, z) + d(z, y) \geq d(x, y)$$

  This is called the *triangle inequality*

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These properties are essential to describing a function which will give us the distance between two points. Any set X with a metric d is called a *metric space*. There can be many metrics on the same space.

The most well-known example is the real line with the absolute value metric. (One can easily see that absolute value satisfies the above properties).

Here are some other examples of sets and metrics.

Example 1. $\mathbb{R}^2$ with $d(u, v) = \sqrt{(a_1 - b_1)^2 + (a_2 + b_2)^2}$ where $u = (a_1, a_2)$ and $v = (b_1, b_2)$.
I realize the notation is new, but the reason for the change will be evident in the next example.
We are enumerating the COORDINATES not the points.

Example 2. We can generalize the above example to any finite dimensional real space. The following metric is called the *usual metric* on $\mathbb{R}^n$.
$\mathbb{R}^n$ and $d(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$

Compute the distance between $(1, 2, 3, 4, 5)$ and $(5, 4, 3, 2, 1)$ in R^5 using the usual metric.
Explain the geometry of the set $B_r(a) = \{x \in X \mid d(x, a) < r\}$. This is the called the *open ball* of radius r about a.

Example 3. We can define a metric on the class of all continuous functions by measuring the upper bound of the distances between two values.
Let $\mathcal{F} = \{f \mid f : X \to X\}$ be the set of functions from X to itself.
Define $d(f, g) = \sup(\{d(f(x), g(x)) \mid x \in X\})$ for each $f, g \in \mathcal{F}$.

The supremum of a set $U \subset X$ is the smallest of the upper bounds and is written

$$\sup(U) = \min\{x \in X \mid x \geq u, \quad \forall u \in U\}$$

Note that X must have a metric in order to define a metric on the function space.

Consider the set of real valued functions on $\mathbb{R}$ with the usual metric.

(1) Compute the distance between the functions $f(x) = 5\sin(x)$ and $g(x) = \cos(x)$.

Solution 1. To compute the distance between these two functions, we must look at the set

$$\begin{aligned} \sup(\{d(f(x), g(x)) \mid x \in \mathbb{R}\} &= \sup(\{|f(x) - g(x)| \mid x \in \mathbb{R}\}) \\ &= \sup(\{|5\sin(x) - \cos(x)| \mid x \in \mathbb{R}\}) \end{aligned}$$

In this case, since $5\sin(x) - \cos(x)$ is bounded, so the maximum is the same as the supremum. We can find the maximum of our new function $h(x) = 5\sin(x) - \cos(x)$ using calculus.

If $h'(x) = 5\cos(x) + \sin(x) = 0$ and the derivative changes from positive to negative, then we have a maximum for h.

$$\begin{aligned} 5\cos(x) + \sin(x) &= 0 \\ 5\cos(x) &= -\sin(x) \\ -5 &= \tan(x) \\ 1.768 &= x \end{aligned}$$

Therefore the maximum for h is at $h(1.768) = 5.10$.

(2) Compute the distance between the functions $f(x) = x^2$ and $g(x) = x^2 + 6$.

(3) Compute the distance between the functions $f(x) = x^3$ and $g(x) = 6x^3 + 2$.

Explain the geometry of the set $B_r(a) = \{x \in X \mid d(x,a) < r\}$

Example 4. The silly metric , AKA "discrete" metric space on a set X is defined by

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

Explain the geometry of the set $B_r(a) = \{x \in X \mid d(x,a) < r\}$.

Example 5. The box metric is another metric which is imposed on finite dimensional real space.
It is given by $d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ takes the sum of the absolute value of the differences between the coordinates.

Find the distance between the points $(1,2,3,4,5)$ and $(5,4,3,2,1)$.
Explain the geometry of the set $B_r(a) = \{x \in X \mid d(x,a) < r\}$.

Why do you think this is called the box metric?

If X, d is a metric space and $x \in X$. We have the following properties:

$$\bigcup_{r>0} B_r(x) = X$$

and

$$\bigcap_{r>0} B_r(x) = x$$

The first of these sets is the union of all open balls containing x, and the second is the intersection of all open balls containing x.

2. Open and Closet Sets in Metric Spaces

Definition 2 (interior point)**.** .
Let X, d be a metric space with metric d, and $Y \subset X$.
We say $a \in Y$ is an *interior point* of Y if there exists $r > 0$ such that $B_r(a) \subset Y$

Let $Y = \{(x, y) \mid \sqrt{x^2 + y^2} \geq 1\}$.
Then $(0, 0)$ is an interior point for Y but $(1, 0)$ is not.
Why?

Definition 3 (interior of Y)**.** .
Let Y be a subset of a metric space X, d. The set of all interior points of Y is called the *interior* of Y and is denoted

$$Y^o$$

. The notation varies depending on the author, but the idea is the same.

Find the interior of Y from the example above.
Let $X \subset \mathbb{R}$ be the image of $(-1, 1)$ under the function $f(x) = x^2$. What is the interior of X?

Definition 4 (open set)**.** .
Let X, d be a metric space.
A set Y is called *open* in X if

$$Y^o = Y$$

it is equal to its' interior.

Give an example of an open set for each of the metrics described in the previous section.

Definition 5 (closed)**.** .
A set is *closed* if it's complement is open.

Give an example of a closed set for each of the metrics described in the previous section.

Definition 6 (limit point)**.** .
Let X, d be a metric space and $Y \subset X$.
We say $a \in X$ is a *limit point* if for all $r > 0$,

$$B_r(a) \cap A \neq \varnothing$$

if all open balls containing a have non-empty intersection with Y.

Definition 7 (boundary point)**.** . We say, $a \in X$ is a *boundary point* of A if and only if for all $r > 0$,

$$B_r(a) \cap Y \neq \varnothing$$
$$B_r(a) \cap Y^c \neq \varnothing$$

every ball of radius, r, about a has nonempty intersection with A and it's complement.

Exercise 1. Show that 1 is a limit point for the set $[0, 1)$ in the usual metric for $\mathbb{R}$.

Exercise 2. Show that $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ is a limit point for the open unit disc in $\mathbb{R}^2$ under the usual metric, but not the box metric.

We are, at least, somewhat familiar with all of the above spaces, and have an idea of the geometry of these spaces. However, these spaces are very restrictive and do not model reality. Although, in space we might think that three dimensional real space is an accurate model, there are too many bends, and anomalies to model space with Euclidean Space. This is where topology comes into play.

Theorems in metric spaces require too much computation. Topological Proofs are much more elegant, concise and have a wider reach.

3. Exercises

Problem 1. Show that in every set is closed and open in the discrete topology.

Problem 2. Show that every point in Y is a limit point of Y.

Problem 3. Let $\bar{Y}$ be the the set of all limit points of Y.
Show that Y is closet if and only if $Y = \bar{Y}$

Problem 4. Show that $(A^o)^o = A^o$.

Problem 5. Show that $\bar{\bar{A}} = \bar{A}$.

Problem 6. Show that the union of open sets is open.

Problem 7. Show that the intersection of closed sets is closed.

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