Introduction to Series

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So, what's a Series?

A series is the sum of the terms in a sequence.

Infinite Series

The infinite series of the sequence (a_n) is written

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

We use the capital Greek letter, Sigma, to denote a sum.
The " n = 1" means we start with the a_1
and add all of the terms up to ∞ which is at the top.

Convergent and Divergent Series

The k-th partial sum of a series $\sum_{n=1}^{\infty} a_n$ is the sum of the first k terms of the sequence:

$$S_k = a_1 + a_2 + a_3 + a_4 + \cdots + a_k$$

We say that the series $\sum_{n=1}^{\infty} a_n$ converges, if the sequence of partial sums, $\lim_{k\to\infty} S_k$ converges.
If S_k diverges, then the series diverges.

Yesterday, you played with the GeoGebra applet I wrote.
From your investigation with GeoGebra, guess whether the series listed below converges or diverges:

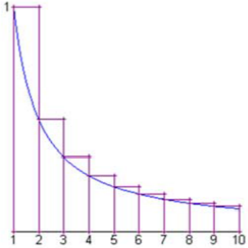
$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \cdots$$

Figure: It took me 5 hours to figure out how to program in GeoGebra, so I'm still proud of it

Divergent: Like the book

Since $\frac{1}{x}$ is a decreasing function on $[1, \infty)$, then the Left Riemann Sum is bigger than the area under the curve.
Since the integral diverges, the series must diverge as well.

$$\begin{aligned} \int_1^{\infty} \frac{1}{x} dx &= \lim_{t \to \infty} \int_1^t \frac{1}{x} dx \\ &= \lim_{t \to \infty} \ln(x)\Big|_1^t \\ &= \lim_{t \to \infty} \ln(t) - \ln(1) \\ &= \infty \end{aligned}$$



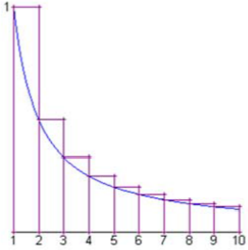
The Left Riemann Sum

This series is called the **The Harmonic Series**.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \cdots$$

diverges because it is the Left Riemann Sum corresponding to

$$\begin{aligned}\int_1^{\infty} \frac{1}{x} dx &= \lim_{t\to\infty} \int_1^{t} \frac{1}{x} dx \\ &= \lim_{t\to\infty} \ln(x)\Big|_1^t \\ &= \lim_{t\to\infty} \ln(t) - \ln(1) \\ &= \infty\end{aligned}$$

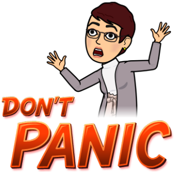


The Left Riemann Sum

What about the other functions?

What did your investigation in GeoGebra say about the sum below? Does it converge or diverge?

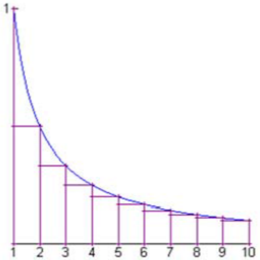
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} \cdots$$



Convergent

Since $\frac{1}{x^2}$ is a decreasing function on $[1, \infty)$, then the **Right** Riemann Sum is smaller than the area under the curve.
Since the integral converges, the series must converge as well.

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}dx &= \lim_{t\to\infty}\int_1^t x^{-2}dx \\ &= \lim_{t\to\infty} -x^{-1}|_1^t \\ &= \lim_{t\to\infty} -\frac{1}{t}+1 \\ &= 1\end{aligned}$$



The Right Riemann Sum

Yes, you guessed it. We have a theorem for this.

Integral Convergence Test

If $f(x)$ is a positive, decreasing, and continuous function for $x > 1$, and $a_n = f(n)$ for all natural numbers n.
Then

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)dx$$

either both converge, or they both diverge.

Check for Understanding 1

Check for Understanding

Use the integral test to decide whether the series below converge or diverge:

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

Check for Understanding 1: Answers

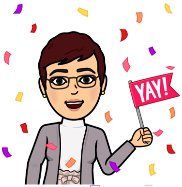
If $f(x) = \dfrac{1}{x^3}$, then it is a decreasing, continuous, positive function on $[1, \infty)$,
and $f(n) = \dfrac{1}{n^3}$. Therefore, we can use the integral test.

$$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_1^t x^{-3}\,dx$$
$$= \lim_{t\to\infty} -\frac{1}{2}x^{-2}\Big|_1^t$$
$$= \lim_{t\to\infty} -\frac{1}{2t^2} + \frac{1}{2}$$
$$= \frac{1}{2}$$

Since the integral converges, then the series,

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges



Check for Understanding 2

Check for Understanding

Use the integral test to decide whether the series below converge or diverge:

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

Check for Understanding 2: Answers

If $f(x) = \dfrac{1}{1+x^2}$, then it is a decreasing, continuous, positive function on $[1, \infty)$,

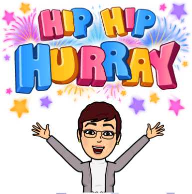
and $f(n) = \dfrac{1}{1+n^2}$. Therefore, we can use the integral test.

$$\begin{aligned}\int_1^\infty \frac{1}{1+x^2}dx &= \lim_{t\to\infty}\int_1^t \frac{1}{1+x^2}dx\\ &= \lim_{t\to\infty} \arctan(x)\Big|_1^t\\ &= \lim_{t\to\infty} \arctan(t) - \arctan(1)\\ &= \frac{\pi}{2} - \frac{\pi}{4}\\ &= \frac{\pi}{4}\end{aligned}$$

Since the integral converges, then the series,

$$\sum_{n=1}^\infty \frac{1}{1+n^2}$$

converges



Check for Understanding 3

Check for Understanding

Use the integral test to decide whether the series below converge or diverge:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Check for Understanding 3: Answers

If $f(x) = \dfrac{1}{\sqrt{x}}$, then it is a decreasing, continuous, positive function on $[1, \infty)$,

and $f(n) = \dfrac{1}{\sqrt{n}}$. Therefore, we can use the integral test.

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_1^t x^{-\frac{1}{2}}\,dx$$
$$= \lim_{t\to\infty} 2x^{\frac{1}{2}}\Big|_1^t$$
$$= \lim_{t\to\infty} 2\sqrt{t} - 2\sqrt{1}$$
$$= \infty$$

Since the integral diverges, then the series,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges