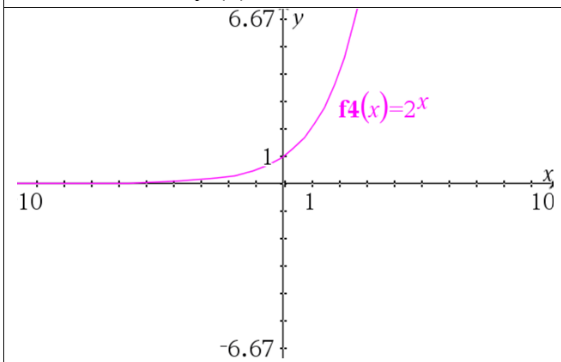
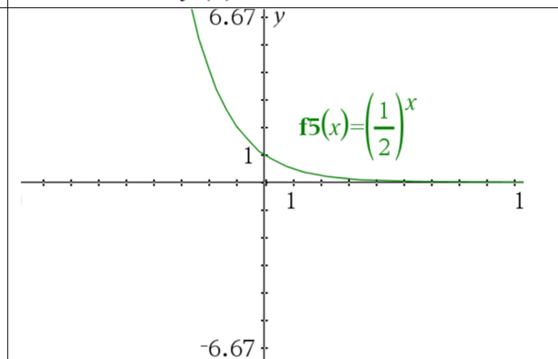
1 Inverse of Exponential Functions

In Chapter 2, we learned that functions which pass the horizontal line test have true inverses. To review, the horizontal line test says that functions whose graph intersect each horizontal line in the range exactly once have true inverses.

We have already learned to graph exponential functions in 4.1,

$f(x)=b^x$ if $b>1$	$f(x)=b^x$ if $0<b<1$
	
Horizontal Asymptote: $y=0$ End Behavior: As $x\to-\infty, y\to 0$	Horizontal Asymptote: $y=0$ End Behavior: As $x\to+\infty, y\to 0$

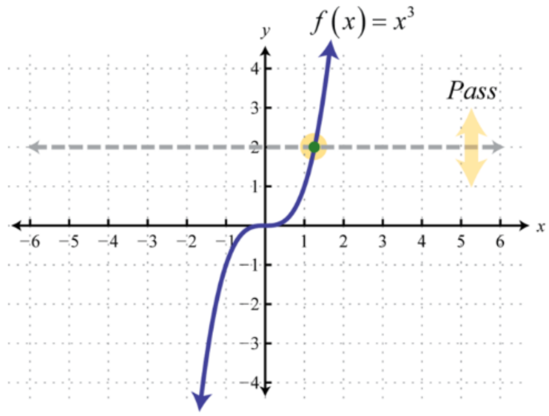
However, it's difficult to tell whether or not the exponential functions pass the horizontal line test because they have a horizontal asymptote.

So, let's decipher what it means to pass the horizontal line test.

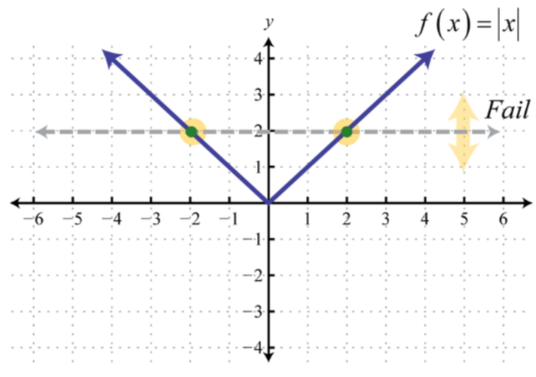
Each horizontal line represents a y-coordinate in the range of the function.

If the graph of the function intersects each horizontal line exactly once, this means that each y-coordinate in the range will be used exactly once.

Therefore, each y-coordinate is associated to one x-coordinate.



One-to-one Function: Yes

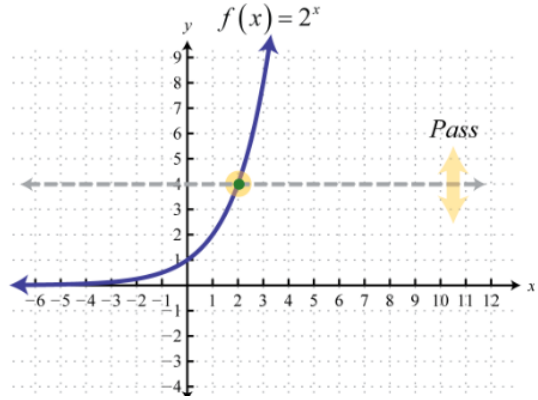


One-to-one Function: No

Notice, the graph on the right has a y-coordinate associated to two different x-coordinates.

The only way this can happen, is if the graph changes direction from increasing to decreasing, or vice-versa.

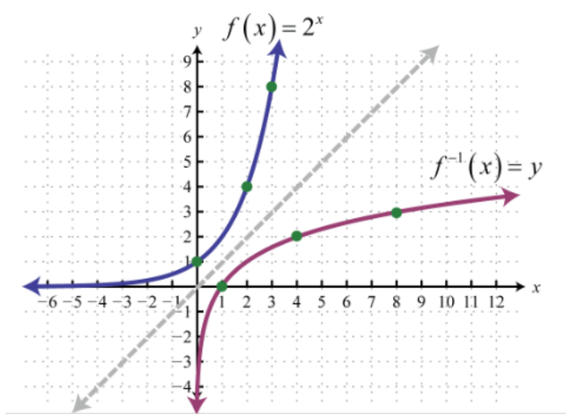
If we look at the graphs of the exponential functions above, we can see that when $b > 1$, the graph is strictly increasing. If $0 < b < 1$, then the graph is strictly decreasing.



This means that all exponential functions will have inverses!

2 Logarithms

So, now that we know that exponential functions have inverses, what is the inverse?
We know that the graph of the inverse function, $g(x)$, is a reflection of the original graph, $f(x)$, over the line $y = x$.



The graph above represents the inverse of the exponential function, $f(x) = 2^x$.
We call that inverse function the **"log base 2"** and it's written $\mathbf{g(x)} = \log_{\mathbf{2}}(\mathbf{x})$.
Since we know the x and y coordinates switch for inverse functions, then:

- the **domain** of $f(x) = 2^x$ is always $(-\infty, \infty)$, then the **range** of $g(x) = \log_2(x)$ is $(-\infty, \infty)$.
- · the **range** of $f(x) = 2^x$ is $(0, \infty)$, then the **domain** of $g(x) = \log_2(x)$ is $(0, \infty)$.
- · the **horizontal asymptote** for $f(x) = 2^x$ at $y = 0$ becomes a **vertical asymptote** for $g(x) = \log_2(x)$ at $x = 0$.

Question 1. Do you think this only holds true for $f(x) = 2^x$, or does it hold for $f(x) = b^x$ for any $b > 0$?

Answer 1. The same domain, range and vertical asymptote would apply to $\log_b(x)$ for any $b > 0$.

The inverse of the function $f(x) = \frac{1}{3}^x$ is $f^{-1}(x) = \log_{\frac{1}{3}}(x)$.
The inverse of the function $f(x) = 10^x$ is $f^{-1} = \log_{10}(x)$.
The inverse of the function $f(x) = e^x$ is $f^{-1} = \log_e(x)$.

Check for Understanding 1. Answer each of the questions below.

(a) What is the domain of $f(x) = \log_{\frac{1}{3}}(x)$?

(b) List all horizontal and vertical asymptotes of $g(x) = 4^x - 1$, if any.

(c) What is the range of $g(x) = (\dfrac{1}{3})^x$?

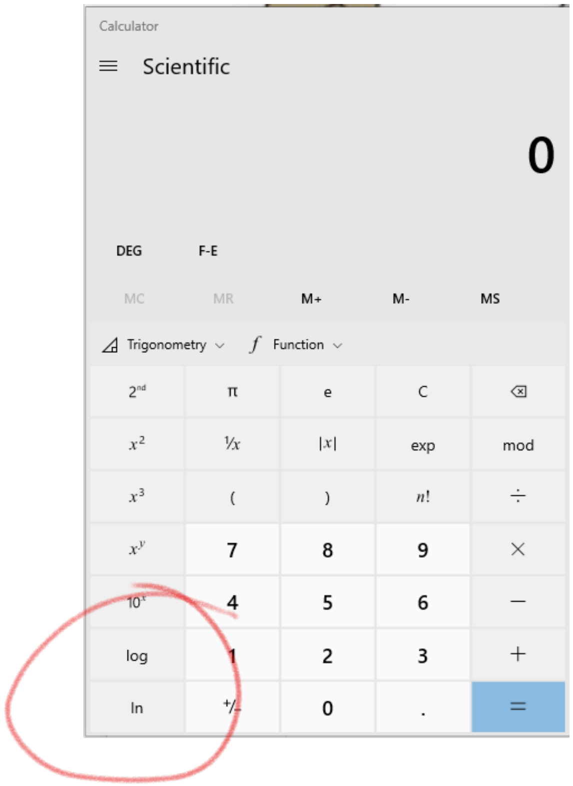
(d) What is the inverse function for $h(x) = 5^x$?

Remark 1. There are two very special log functions:

$$\log_{10}(x) = \log(x) \text{ is called the } \textbf{common log}$$
$$\log_e(x) = \ln(x) \text{ is called the } \textbf{natural log}$$

The mathematician, Leonhard Euler , was a Swiss mathematician who was the creator or discoverer of e^x and its inverse the natural log in 1741.

Since he spoke French, he called it "le logarithme naturel", which is why it is thought that Irving Stringham, a professor of University of California, Berkely, coined the notation in 1893.
If you look at your calculator, you normally see two buttons with these labels.



Check for Understanding 2. Use your calculator to find these values

(a) log(1234)

(b) ln(e)

(c) log(153)

(d) ln(1)

(e) log(156842)

How many digits are the numbers in (a), (c), and (e).?
What is the relationship between the number of digits in the numbers and the log of the number?

Test your hypothesis by trying the log of more numbers.

Answers for Check for Understanding 1:

(a) $(0, \infty)$ (b) horizontal asymptote: $y = -1$ (c) $(0, \infty)$ (d) $\log_5(x)$

Answers for Check for Understanding 2:

(a) 3.091 (b) 1 (c) 2.185 (d) 0 (e) 5.195