Combinatorial and Computational Number Theory

You may be familiar with the concept of factorial as a way to count the number of ways to rearrange a set number of objects. Algebraically, we define *n* factorial, denoted *n*!, as the product of all the positive natural numbers less than or equal *n*.

$$n! = n\cdot(n-1)\cdot(n-2)\cdot(n-3)\ \dots\ 3\cdot2\cdot1$$

Combinatorically, we define *n* factorial to be the number of ways to rearrange *n* objects. Using this definition, it makes sense that 0! = 1, because there is one way to rearrange zero objects.

This is useful in counting the number of seating arrangements one can create for a set number of people. For example, using only factorial, one can now answer the question on everyone's mind;

How many ways can Mrs. Bailey rearrange our class if she only uses the first 10 chairs?

_____ _____ _____ _____ _____ _____ _____ _____ _____ _____
1st 2nd 3rd 4th 5th 6th 7th 8th 9th 10th

Unfortunately, Mrs. Bailey will not restrict the seating arrangements to the first ten chairs. She wants to use any of the 32 chairs available in the classroom. Now how many ways can Mrs. Bailey rearrange her 10 students in Introduction to Abstract Math/ Category Theory?

_____ _____ _____ _____ _____ _____ _____ _____ _____ _____
1st 2nd 3rd 4th 5th 6th 7th 8th 9th 10th

For the first student, there are 32 chairs from which to choose, for the second student, there are only 31 chairs from which to choose, etc, etc.. This is called a **10-*permutation***. We are permuting 10 people in a set of 32 chairs, so the number can be modeled by 32 permute 10, and is denoted

$$P\begin{pmatrix}32\\10\end{pmatrix} \quad \text{or} \quad P(32,10) \quad \text{or} \quad {}_{32}P_{10}$$

Furthermore, the number of rearrangements of the 10 students in the 32 chairs can also be given by

$$\frac{32!}{32-10!}$$

Permutation: An *r*-permutation of a set *S* of *n*-objects is an ordered *r-tuple* from a set *S*. Then number of *r*-permutations in a set, *S,* of cardinality *n* is

$$\frac{n!}{(n-r)!} = P\begin{pmatrix}n\\r\end{pmatrix}$$

Example 1: The 97 students in band are competing from prizes at the end of the year. How many ways can students win first, second and third prize?

Solution: *We are looking for the number of 3-permutations in the set of cardinality 97. Therefore, there are* ${}_{97}P_3 = (97)(96)(95) = 884640$

Example 2: Define a *word* to be a string of letters from the 26-letter alphabet. How many 4-letter words are there?

However, permutations alone are unhelpful when considering the number of 4-letter collections from the alphabet, because words require that we take into consideration the order of the elements, whereas collections require us to ignore the order of the elements.

<u>Combination:</u> An *r*-combination of a set *S* of cardinality *n* is a subset in *S* of cardinality *r*,
Denote the set of all subsets of cardinality *r* in a set *S* as $\mathfrak{p}_r(S)$. It is clear that $\mathfrak{p}_r(S) \subset \mathfrak{P}(S)$, is a subset of the power set of *S*. The cardinality of $\mathfrak{p}_r(S)$ is

$$\frac{n!}{(n-r)!r!} = C\begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n \\ r \end{pmatrix} = {}_nC_r = C(n,r)$$

By dividing $\frac{n!}{(n-r)!} = P\begin{pmatrix} n \\ r \end{pmatrix}$ by the *r*!, we are dividing by the number of rearrangements of the elements in *r*-tuples that contain the same set of elements, thereby, getting rid of all the n-tuples we counted as distinct that consisted of elements of the same subset of *S*.

Lemma: Given any natural number *n* and any positive natural number $r \le n$, $C\begin{pmatrix} n \\ r \end{pmatrix} | P\begin{pmatrix} n \\ r \end{pmatrix}$

Proof: To prove this statement, we need only look at the definition of

$$\frac{n!}{(n-r)!} = P\begin{pmatrix} n \\ r \end{pmatrix}$$

and the definition of

$$\frac{n!}{(n-r)!r!} = C\begin{pmatrix} n \\ r \end{pmatrix}$$

to see that

$$C\begin{pmatrix} n \\ r \end{pmatrix} r! = P\begin{pmatrix} n \\ r \end{pmatrix}$$

□

Example 3: Suppose the band of 97 students no longer wanted to award first, second and third prize, but instead, wanted to honor to top three students in band. How many different ways could 3 students be chosen from a band of 97 students?

Solution: We are looking for the cardinality of $\mathfrak{p}_3(B)$, where *B* is the set of all band students. By definition, this is $\frac{97!}{(97-3)!3!}$ =147440.

Example 4: How many different hands of 5 cards are possible from a deck of 52 cards?

Example 5: Show that the product of any *n* consecutive integers is divisible by the product of the first *n* integers.

Exercises:

1) Let S be a set of cardinality n. Let $\mathfrak{P}(S)$ be the power set of S. Show that the cardinality of the power set of S, $|\mathfrak{P}(S)| = 2^n$.
2) Show that
$$\sum_{k=0}^{n} C\binom{n}{k} = 2^n$$
3) Prove that for any $n \in \mathbb{N}$, and any $r \le n$, such that $r \in \mathbb{N}$
$$C\binom{n}{r} = C\binom{n-1}{r} + C\binom{n-1}{r-1}$$
4) Show that
$$\sum_{k=0}^{n/2} C\binom{n}{2k} = 2^{n-1} = \sum_{k=0}^{n/2} C\binom{n}{2k+1}$$
5) Show that
$$C\binom{n}{r} = C\binom{n}{n-r}$$
6) Prove that if p is a prime number, and $0 < a < p$, then $p \mid C\binom{p}{a}$
7) ***Binomial Theorem:*** Use the principle of mathematical induction to show that
$$(x+y)^n = \sum_{k=0}^{n} C \ge \binom{n}{k} x^k y^{n-k}$$
8) ***Fermat's Little Theorem:*** Use mathematical induction, and the last two exercises to show that $p \mid n^p - n$
9) Prove that p is the smallest prime number that divides $(p-1)!+1$
10) Find the probability of achieving the following in a deck of 52 cards.
    a. Full house
    b. Straight
    c. Straight flush
    d. Flush
    e. Poker
    f. 3 of a kind
    g. 2 pairs
    h. Royal Flush