Introduction to Number Theory

The Peano axioms were created to prove that the ZFC model of set theory implies the existence of the natural numbers. For example, in one of the axioms, it states that the empty set is a set.

$$\varnothing$$
$$\{\varnothing\}$$
$$\{\{\varnothing\}\}$$

Another axiom in the ZFC model of set theory states;
If *A* is a set, then the set containing *A* is also a set.

With the two axioms above we can create a countably infinite in which the empty set corresponds to zero, the set containing the empty set corresponds to 1, etc. etc.

<u>Peano Axioms:</u>

Let *P*(*x*) be a predicate statement, whose universe of discourse is the natural numbers. Suppose the following are true statements.

1) **<u>Base Case:</u>** *P*(1) : *the statement is true for n = 1*
2) **<u>Induction Hypothesis:</u>** *P*(*n*) implies *P*(*n*+1): *the statement being true for n implies the statement is true for n +1*

   If the two statements above are true, then *P*(*k*) is true for all $k \in \mathbb{N}$.

Let's prove that $1+2+3+\ldots+n=\frac{(n)(n+1)}{2}$ using the Peano axioms. This type of proof is called ***mathematical induction.***

Proof: First we must show the statement is true for *n* = 1, *P*(1) is a true statement.

$$1=\frac{1(1+1)}{2}=\frac{2}{2}$$

Now we need to show that *P*(*n*) implies *P*(*n*+1).

Supposing *P*(*n*) is true, we would have that

$$(*) \qquad 1+2+3+\ldots+n=\frac{(n)(n+1)}{2}$$

We need to show that

$$1+2+3+\ldots+n+(n+1)=\frac{(n+1)(n+2)}{2}$$

In order to do this, we will take the equation (*) and add *n* +1 to the left hand side.

Then

$$1+2+3+\ldots+n+(n+1)=(1+2+3+\ldots+n)+(n+1)$$

By induction hypothesis,

$$=\frac{(n)(n+1)}{2}+(n+1)$$

$$=\frac{(n)(n+1)}{2}+\frac{2(n+1)}{2}$$

$$=\frac{(n)(n+1)+(2)(n+1)}{2}$$

$$=\frac{(n+2)(n+1)}{2}$$

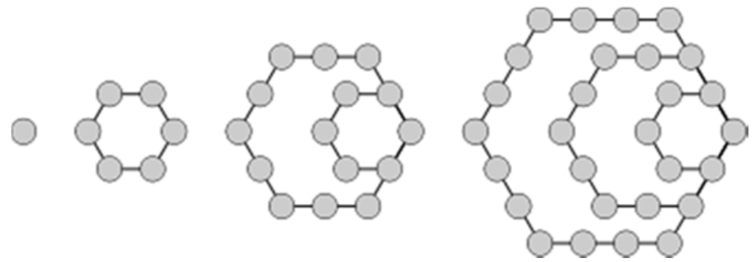
So then we have that P(n) implies P(n+1). Therefore, since, using the base case, we proved that P(1) is true. By the induction hypothesis, since P(1) is true, then P(2) is true. Repeating this induction hypothesis, we get that P(2) is true, implies P(3) is true. Following this to its inevitable conclusion, we find that the statement is true for all natural numbers.

Examples in Class:

1.Show $\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$ is true for all n $\in \mathbb{N}$

2.Show $1^2+2^2+\ldots+n^2=\frac{(n)(n+1)(2n+1)}{6}$ for all n $\in \mathbb{N}$.

Example 1: Hexagonal Numbers



1st 2nd 3rd 4th

The sequence is defined by the number of dots in the hexagon and on its perimeter.

1st term : 1
2nd term: 6 = 1 + 5
3rd term: 15 = 6 + 9
4th term: 28 = 15 + 13
5th term: 28 + ___

nth term : (n-1)th term + ______

The recursive sequence is defined to be

$$H_1 = 1$$
$$H_n = H_{n-1} + \_\_\_\_\_\_\_$$

Example 2: Triangular Numbers

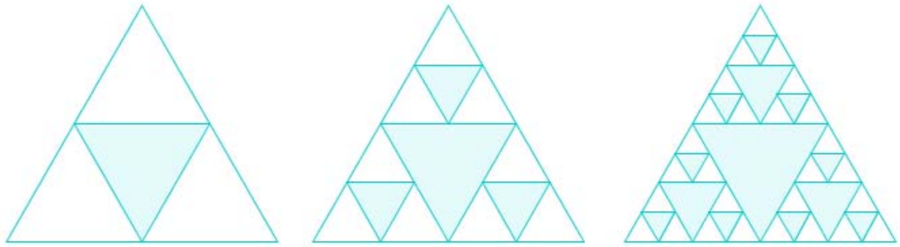
1st term : 1
2nd term: 3 = 1 +
3rd term: 6 = 3 +
4th term: 10 = 6 +
5th term: 15 = 10 + ___

nth term : (n-1)th term + ______

The recursive sequence is defined to be

$$H_1 = 1$$
$$H_n = H_{n-1} + \_\_\_\_\_\_\_$$

Example 3: The Seriepinski Triangles



The number of blue triangles:

1st term:
2nd term:
3rd term:

Recursive Definition:

The number of white triangles:

1st term:
2nd term:
3rd term:

Nth term:

Recursive Definition:

The length of the sides of the smallest triangles after nth iteration:

1st term:
2nd term:
3rd term:

Nth term:

Recursive Definition:

The total area of the blue triangles:

1st term:
2nd term:
3rd term:

Nth term:

Recursive Definition: