Lines and Circles on the Complex Plane

Lines

If I asked you for the equation whose locus is a line, you would probably give me

$$y = mx + b$$

Unfortunately, in the complex plane, you would be saying that $y = \text{Im}(z)$, the imaginary part of *z*, and $x = \text{Re}(z)$, the real part of *z*. So, in order to consider an equation whose locus is a line in the complex plane, you must substitute

$$\text{Im}(z) = m(\text{Re}(z)) + b$$

where *m* and *b* are real numbers. However, we would like to write this equation in terms of the actual complex number, *z*. So how could we do that?

Talk to your neighbor and discuss how we could achieve the imaginary part of *z* and the real part of *z*, using *z*. I'll give you a hint: use z and $\overline{z}$.

Re(*z*) =

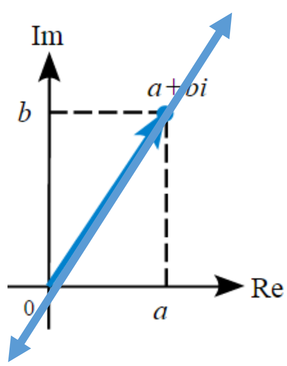
Im(*z*) =

Good job! Now substitute this into the equation we have above.

Equation for a line in the complex plane:

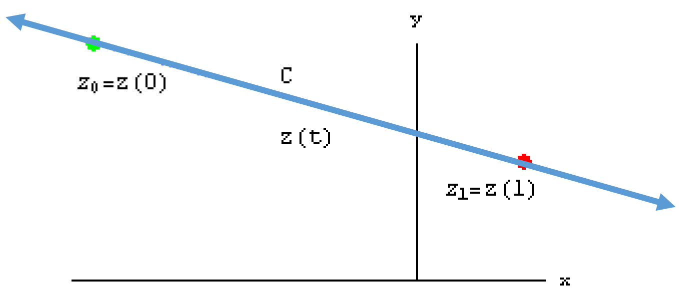
However, this isn't that pretty. Although this equation may be helpful later, let's see if we can find a different form.

Consider the equation whose locus is the line below. These are all scalar multiples of *a* + *ib*.



If you don't believe me, let's look at the scalar multiples: $\{(a+ib)t \mid t \in \mathbb{R}\}$. Let *x* be the real part of this set of complex numbers and *y* be the imaginary part. What do you get?

Now let us look at a vector equation for a line through z_0 and z_1.



Using only z_0 and z_1, how can we characterize the direction of this line?

I'll start you off: Let $z_0 = a + ib$ and $z_1 = c + id$.

If you still need help, a good strategy is to plug in numbers for *a*,*b*,*c* and *d* and figure it out with that example.

$z_0 = 2 + 3i$ and $z_1 = 5 - 3i$

Then graph the line through those two points on your white board. You can draw the direction. Now try to figure out if you can get that slope by doing something to z_0 and z_1.

If you think your algorithm will work for other complex numbers, test it out. Try some other two random numbers, then count the slope and use your algorithm. If it works again, chances are you're right! But you still don't KNOW it will work all the time. So figure out exactly why that works, then if you are convinced your method works for any line through two points, convince your neighbor. If they are thoroughly convinced, you have a proof! Good for you.

However, a direction is not sufficient to get an equation of a line through z_0 and z_1, because a slope will only get you a line through the origin. To finish you must translate your line to z_0 or z_1.

Write the equation of a line through z_0 and z_1 here:

Circle

The definition of a circle is a set of points on a plane equidistant to a fixed point.

This makes finding an equation whose locus is a circle very easy because the complex distance from z to z_0 is given by

$$\begin{aligned} |z-z_0|^2 &= (z-z_0)\overline{(z-z_o)} \\ &= [(x-x_0)+i(y-y_0)][(x-x_o)-i(y-y_o)] \\ &= (x-x_0)^2+(y-y_0)^2 \end{aligned}$$

So the equation of a circle is

$$|z-z_0|^2 = r^2$$

Now write the equation in terms of z and $\overline{z}$.

Exercises:

1) Write the equation of the line through $-2+5i$ and $7-3i$ in two different ways.

2) Which of the following equations represents a line? Graph it.

$$(1+5i)z+(1+5\mathrm{i})\overline{z}+3i=0$$

$$(1+5i)z+(1-5i)\overline{z}-3=0$$

3) Write the equation of the circle centered at 3 – 2i of radius 4 in two different ways.