Generalized Binomial Theorem

Although Chinese mathematician Yang Hiu knew of the pattern in the coefficients

$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

It was Pascal who came up with the idea of the triangle that would easily compute the coefficients of this binomial expansion.

However, as one can see, although Pascal's Triangle is useful for the expansion of powers of $(a + b)$ from 1 to 25, writing over 25 rows of this coefficient triangle would be tedious and time-consuming. Newton wanted to find another way.

For coefficients in the Natural Numbers, i.e. {1, 2, 3, 4, 5,….}, it is known that

$$(a+b)^n = C\binom{n}{n}a^nb^0 + C\binom{n}{n-1}a^{n-1}b^1 + C\binom{n}{n-2}a^{n-2}b^2 + \cdots + C\binom{n}{n-k}a^{n-k}b^k + \cdots C\binom{n}{0}a^0b^n$$

where $C\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

In Journey Through Genius, Newton wrote of his attempt to generalize the expansion of $(a + b)$ with rational exponents:

> Newton's version of his binomial expansion is presented here as he explained it in a significant 1676 letter to his great contemporary Gottfried Wilhelm Leibniz (a letter delivered via Henry Oldenberg of the Royal Society). Newton wrote:
>
> $$(P+PQ)^{m/n} = P^{m/n} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \cdots$$
>
> where $P+PQ$ is the binomial to be considered; where m/n is the power to which we shall raise the binomial "whether that power is integral or (so to speak) fractional, whether positive or negative"; and where A, B, C, and so on represent the immediately preceding terms in the expansion.

The last sentence means that $A = P^{\frac{m}{n}}$ and $B = \dfrac{m}{n}A = \dfrac{m}{n}P^{\frac{m}{n}}$.

Exercise 1: Write C and D in terms of P, m and n. Then re-write the entire sum with coefficients which only use m, n and P.

Looking at the previous simplification of Newton's Sum, you may note that $\frac{m-n}{2n}=\frac{\frac{m}{n}(\frac{m}{n}-1)}{2}$.

Exercise 2: Rewrite your summation from Exercise 1 with this way of writing the coefficients.

Does this look like something? If $C\binom{n}{k}=\frac{n!}{k!(n-k)!}=\frac{n(n-1)(n-2)\cdots(n-(k+1))}{k!}$ what would this be if we put a fraction in for *n*.

$$C\begin{pmatrix}\frac{m}{n}\\2\end{pmatrix}=\frac{\frac{m}{n}(\frac{m}{n}-1)}{2!}$$

Newton really wanted to show that

$$(x+y)^n=\sum_{k=0}^{\infty}C\binom{n}{k}x^k y^{n-k}$$

where $C\binom{n}{k}=\frac{(n)(n-1)(n-2)(n-3)\cdots(n-k+1)}{k!}$ and *n* is a **real number**.

Exercise 3: a. Using the generalized binomial theorem, find the first five terms of the expansion $\frac{1}{(1+x)^3}$

Recall that in AP Calculus AB, we learned that if *f* was differentiable in an interval containing 0, then

$$f(x)=\frac{f(0)}{0!}x^0+\frac{f'(0)}{1!}x^1+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\cdots\frac{f^{(n)}(0)}{n!}x^n+\cdots$$

b. Find the first five terms of this MacLaurin series expansion. Does it match with your expansion from part (a)?

c. Newton checked the expansion by multiplying the expression in part (a) with $(1+x)^3$ to get 1. Multiply your expression in part (a) with $(1+x)^3$ and find the first 6 terms.

Exercise 4: Consider the function $f(x)=(1-x)^{-5}$

(a) Use the generalized binomial theorem to find the expansion for $f(x)$.

(b) Use MacLaurin series to find the generalized expansion aroung $f(x)$. Are they the same?

Exercise 4: Consider the function $f(x)=\sqrt{1-x}$

(a) Use the generalized binomial theorem to find the expansion for $f(x)$.

(b) Use MacLaurin series to find the generalized expansion aroung $f(x)$. Are they the same?

Exercise 5: Newton used his expression for $f(x)=\sqrt{1-x}$ to approximate square roots. For example, he Approximated $\sqrt{7}=\sqrt{9-2}=\sqrt{9(1-\frac{2}{7})}=3\sqrt{1-\frac{2}{7}}=3(1-\frac{1}{9}-\frac{1}{162}-\frac{1}{1458}-\frac{5}{5248}-\frac{7}{472392}\cdots)$

Try using this method to approximate $\sqrt{13}$ with two terms.