Normal Subgroups and Quotient Groups

Marizza Bailey

1 Cosets

We have now noticed that if $g : G \to H$ is a group homomorphism from a group G into H, then the elements of the image are in one-to-one correspondence with cosets of the kernel. Let us review the definition of coset. To be more general, we will use multiplication for the operation on G.

Definition 1 (left coset)**.** .
Let G be a group.
Let $g \in G$.
Let $K \leq G$ be a subgroup of G.
Then a set defined by

$$gK = \{gk \mid k \in K\}$$

is a *left coset* of K in G.

Similarly, a *right coset* of K in G is defined to be a set of the form

$$Kg = \{kg \mid k \in K\}$$

I claim that the relation defined by $g_1 \equiv g_2$ if and only if $g_1 \in g_2K$ (they are in the same LEFT coset) is an equivalence relation.

Proof. In order to prove this is an equivalence relation, we must prove,

symmetry
Since $K \leq G$ is a subgroup of G, then it contains the identity, $1 \in K$.
Let $g \in G$. Then $g = g \cdot 1 \in gK$ implies $g \equiv g$.

reflexivity
Let $g, h \in G$ Suppose $g \equiv h$. Then $g \in hK$ implies there exists $k \in K$ such that $g = hk$.
Then $gk^{-1} = h$.
Since K is a subgroup, it contains the inverse of k. Furthermore, $k^{-1} \in K$ implies $gk^{-1} = h \in gK$.
Therefore $h \equiv g$

transitivity
Let $g, h, t \in G$.
Suppose $g \equiv h$ and $h \equiv t$.
Then by definition of $g \equiv toh$, $g \in hK$ or there exists $k_1 \in K$ such that $g = hk_1[1]$
Also, by definition, there exists $k_2 \in K$ such that $h = tk_2[2]$.
Recall, we would like to show that $g \equiv t$.
By substituting equation [2] $h = tk_2$ into equation [1] $g = hk_1$ we achieve

$$g = tk_2k_1$$

By the closure property of subgroup, we know $k_1k_2 \in K$.
Therefore, $g = tk_1k_2 \in tK$ implies $g \equiv t$. □

A similar proof shows that being the the same right coset is also an equivalence relation.

It is useful to think of cosets as translations of subgroups in the original group. The cardinality of each coset is the same, and therefore the number of cosets and the order of the subgroup must divide the original group. We call the number of cosets of a subgroup H in G, the *index of the subgroup, and denote it by*

$$[G : H]$$

This leads us to our next MAJORLY HUGE THEOREM

Theorem 1 (MAJORLY HUGE THEOREM). *(just kidding it's actually LaGrange's Theorem)*
Let $H \leq G$ and the index $o\ f\ H$ in G is finite, then

$$|H|[G : H] = |G|$$

In words, this theorem states that the product of the order of the subgroup H and the index of H in G is equal to the order of G.

2 Normal Subgroups

Now the real question is:

When is the set of equivalence classes a GROUP?

Suppose $H \leq G$ *and define* $G/H = \{H, g_1H, g_2H, \ldots, g_nH\}$ *In order to check whether or not* G/H *is a group we must first define an operation.*

$$gH * kH = gkH$$

By multiplying both sides of the equation by g^{-1} *on the left, we see this is equivalent to*

$$g^{-1}gH * kH = g^{-1}gkH$$

$$HkH = kH$$

If $kH = Hk$ *then the equality will hold. However, this is not always the case.*

Example 1. Let $\tau \in D_4$ be the reflection in the Dihedral group of the square defined by $\tau = (13)$.
Let $H =< \tau >$ be the subgroup generated by τ. It has order 2 and contains only τ and the identity.
Let $\rho = (1234)$ be the rotation in D_4 with order 4.
Then

$$\rho\tau = (1234)(13) = (12)(34)$$
$$\tau\rho = (13)(1234) = (14)(23)$$

Therefore, the coset $\rho H \neq H\rho$.

Factor Groups and Normal Subgroups

Let σ = (13) be a reflection in D_4 and H = <ρ> be the subgroup generated by È.

Let ρ= (1234) be a rotation in D_4.

Note that σH ≠ Hσ

Let us look at the cosets of H in D_4.

D_4

H = { (13) , e}
σH = { (12)(34), (1234) } = $H\sigma^3$
σ^2H = { (24), (13)(24) } = $H\sigma^2$
σ^3H = { (14)(32), (1432) } = Hσ

To see if the set of cosets form a group, we must see if

gHkH = gkH