"To calculate the total mass of a wire lying along a curve in space … we need a more general notion of integral" (Thomas, p.901) Integrals along a curve are called *path integrals* or *line integrals.* If you recall, a Complex-plane curve is defined to be a function in one variable whose image lives in the complex plane.

$$r(t) = x(t) + iy(t)$$

If *f* is a complex-valued continuous function, then

$$f(x+iy) = w$$

then the composition of *f* and *r*

$$f(r(t)) = f(x(t)+iy(t))$$

is a continuous, real-valued single variable function.

The curve is partitioned into small arcs given by the image of a partition on the closed interval on which the path is defined. For example, suppose $r:[a,b] \to \mathbb{C}$, then we can partition [*a*, *b*] into subintervals

$$0 = t_0 < t_1 < t_2 < t_3 < \ldots < t_{n-1} < t_n = 1$$

The image of these points gives a partition of the curve, *r*,

$$r(0) = r(t_0), r(t_1), r(t_2), r(t_3), \ldots, r(t_{n-1}), r(t_n) = r(1)$$

yielding the Riemann Sum,

$$S_n = \sum_{k=1}^{n} f(x_k + iy_k)\Delta s_k$$

Letting *n* go to infinity,

DEFINITION: If *f* is defined in a region containing the curve *C* given parametrically by $r(t) = x(t) + iy(t)$, then the line integral of *f* over *C* is

$$\int_C f(z)ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k + iy_k)\Delta s_k$$

Evaluating a path integral is very easy. Just compose and change the limits of integration

How to evaluate a line integral:

$$\int_C f(z)ds = \int_a^b f(r(t))\,|v(t)|\,dt$$

where $v(t) = r'(t)$ and $s(t) = |v(t)|$.

Why do we have that extra $|v(t)|\,dt$ ?

Because we are integrating over the curve and, therefore, the *ds* refers to the arclength differential.

Since $\frac{ds}{dt} = |v(t)|$ then $ds = |v(t)|\,dt$, so when we change the variable from *s* to *t*, we need to multiply by the change in area.

Here is an example of path integration in over the reals.

**EXAMPLE 1** Integrate $f(x, y, z) = x - 3y^2 + z$ over the line segment $C$ joining the origin to the point (1, 1, 1) (Figure 16.2).

**Solution** We choose the simplest parametrization we can think of:

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \qquad 0 \le t \le 1.$$

The components have continuous first derivatives and $|\mathbf{v}(t)| = |\mathbf{i} + \mathbf{j} + \mathbf{k}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ is never 0, so the parametrization is smooth. The integral of $f$ over $C$ is

$$\int_C f(x, y, z)\, ds = \int_0^1 f(t, t, t)\left(\sqrt{3}\right) dt \qquad \text{Eq. (2)}$$

$$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\, dt$$

$$= \sqrt{3}\int_0^1 (2t - 3t^2)\, dt = \sqrt{3}\left[t^2 - t^3\right]_0^1 = 0. \qquad \blacksquare$$

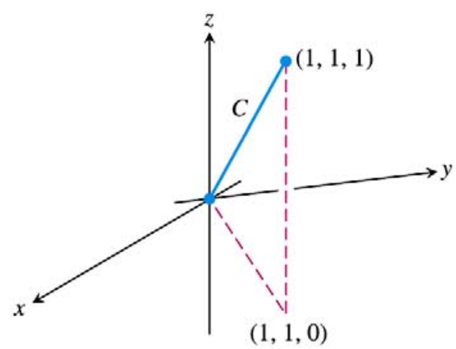


**FIGURE 16.2** The integration path in Example 1.

> The value of the line integral along a path joining two points can change if you change the path between them.

In other words, the integral is completely dependent on the path through the two points.

An example from Churchill in the Complex Numbers

## 38. DEFINITE INTEGRALS OF FUNCTIONS $w(t)$

When $w(t)$ is a complex-valued function of a real variable $t$ and is written

$$w(t) = u(t) + iv(t), \tag{1}$$

where $u$ and $v$ are real-valued, the definite integral of $w(t)$ over an interval $a \le t \le b$ is defined as

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt, \tag{2}$$

provided the individual integrals on the right exist. Thus

$$\operatorname{Re}\int_a^b w(t)\,dt = \int_a^b \operatorname{Re}[w(t)]\,dt \quad \text{and} \quad \operatorname{Im}\int_a^b w(t)\,dt = \int_a^b \operatorname{Im}[w(t)]\,dt. \tag{3}$$

**EXAMPLE 1.** For an illustration of definition (2),

$$\int_0^1 (1+it)^2\,dt = \int_0^1 (1-t^2)\,dt + i\int_0^1 2t\,dt = \frac{2}{3} + i.$$