Base Representation

Any natural number can be written base 10. We take that for granted because that is how we learned the number system.

For example,

$$209 = 2(10^2) + 0(10^1) + 0(10^0)$$

Or we can be more elegant and write it as

$$209 = \sum_{i=0}^{2} a_i 10^i \text{ where } a_0 = 9, a_1 = 0, \text{ and } a_2 = 2$$

In fact, any natural number can be written as a finite series in **base 10** whose coefficients are between 0 and 9.

i.e., For all $k \in \mathbb{N}$, there exists

$$k = \sum_{i=1}^{n} a_i 10^i \text{ such that } a_i \in \mathbb{N} \text{ and } 0 \le a_i \le 9$$

But why do we need to use 10 all the time. Is not a number by any other base, value just as sweet?

For example, we can write 209 in **base 2** as

$$209 = 1(2^7) + 1(2^6) + 0(2^5) + 1(2^4) + 0(2^3) + 0(2^2) + 0(2^1) + 1(2^0)$$

$$209 = 1101001$$

In base 2 all coefficients of the powers of 2 must be between 0 and 1, inclusive.

Try writing the following numbers in base 2, or binary.

1. 14
2. 256
3. 117

However, we can represent a natural number, n, in any base, b. The representation would be of the form,

$$k = \sum_{i=1}^{n} a_i b^i, \text{ where } a_i \in \mathbb{N} \text{ such that } 0 \le a_i \le b - 1$$

Try writing the following number in the base indicated.

1. 163 base 3
2. 163 base 2
3. 163 base 5

Problems:

1. Write the relationship between a binary integer and one twice as large.

2. How many zeros would 58! have if it were written in base 12?
   a. Find the exponent of 2 in 58!
   b. Find the exponent of 3 in 58!
   c. Find the largest power of 12 that divides 58!
3. Write 2671_9 in base 3. Generalize your method to one for converting from any perfect square base to a base representation in its square root.
4. How many base 6 integers less than 1000_6 include 5 as a digit?

Challenge:

(*Mandlebrot)* What is the smallest integer $b > 3$ for which the base *b* number 23_b is a perfect square?

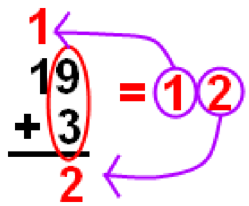
(*AMC*) A faulty car odometer proceeds from digit 3 to digit 5, always skipping the digit 4, regardless of position. For example, after traveling one mile the odometer changed from 000039 to 000050. If the odometer now reads 002005, how many miles has the car actually traveled?

(*AIME*) A cryptographer devises the following method for encoding positive integers. First the integer is expressed in base 5. Second, a 1-1 correspondence is established between the digits that appear in the expressions in base 5 and the elements of the set *V*, *W*, *X*, *Y*, and *Z*. Using this correpondence, the cryptographer finds that three consecutive integers in increasing order are coded *VYZ*, *VYX*, *VVW* respectively. What is the base-10 expression for the integere coded as *XYZ?*

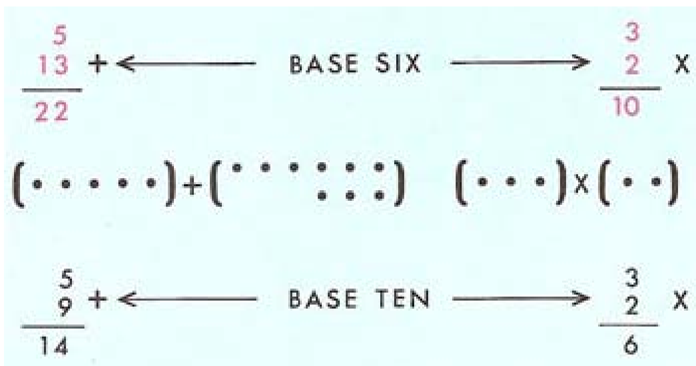
Base Number Arithmetic

Base Addition:

When adding in different bases, the only difference is that we have to carry a one at a different instance. For example, in base 10, we would always carry the tens digit when the sum of two numbers was over 9.

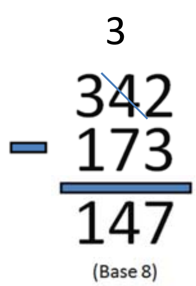


However, in another base, we will carry whenever the sum of two number is greater than or equal to the base.



Base subtraction:

Base subtraction is essentially the same except we are borrowing, not ten, but whatever the base is.



If you notice in the example above, we cannot subtract 2 from 3. Normally, we would have to borrow a ten from the next digit in the top number, but in this case, we are borrowing an 8. Then 8+2 = 10 and 10 -3 = 7, yielding the units digit of our answer. Now the "8's" digit is decreased by one making it a 3. Finish the rest of the borrowing process and see if you get the same answer.

Now convert both 342_8 and 173_8 to base 10 digits and subtract them. Is your answer equivalent to 147_8?

Base Number Multiplication:

How would you multiply $34_6 \times 241_6$?

Here is a deconstruction of the process that might help you.

$$\begin{aligned}
34_6 \times 241_6 &= (3\cdot6^1 + 4\cdot6^0)\times(2\cdot6^2 + 4\cdot6^1 + 1\cdot6^0) \\
&= 3\cdot6^1 \times (2\cdot6^2 + 4\cdot6^1 + 1\cdot6^0) + 4(2\cdot6^2 + 4\cdot6^1 + 1\cdot6^0) \\
&= 6^4 + \cdot2\cdot6\cdot6^2 + 3\cdot6^1 + (6+2)6^2 + (2\cdot6+4)6^1 + 4\cdot6^0 \\
&= 6^4 + 2\cdot6^3 + 3\cdot6^1 + 6^3 + 2\cdot6^2 + 2\cdot6^2 + 4\cdot6^1 + 4\cdot6^0 \\
&= 1(6^4) + 3(6^3) + 4(6^2) + 1(6^2) + 1(6^1) + 4(6^0) \\
&= 13514_6
\end{aligned}$$

However, as you can see, this would be an inefficient way to multiply. So how can we multiply vertically?

$$\begin{array}{r}
{}^{2}\,{}^{2}\quad \\
241_6 \\
\times \quad 34_6 \\
\hline
1444_6 \\
12030_6 \\
\hline
13514_6
\end{array}$$

To make my life easier, I will denote numbers in base 10 with parenthesis. So in base 6, 10_6 (6).

First row: The first digit is easy. 1 times 4 is 4, but 4 times 4 (16) is (12) + 4 which is really 24_6. So when we carry the 2, we are left with a 4 for the next digit in the product, and 2 times 4 is 12_6(8) and when we carry the two we get 14_6(10), resulting in the last two digits in the first row product.

Second row: We still add a zero marker since the next row is one 6 higher than the previous row. See if you can explain the rest of the digits in this row.