Combinations

You may be familiar with the concept of factorial as a way to count the number of ways to rearrange a set number of objects..

$$n! = n\cdot(n-1)\cdot(n-2)\cdot(n-3)\ \ldots\ 3\cdot2\cdot1$$

Combinatorically, we define *n* factorial to be the number of ways to rearrange *n* objects.

This is useful in counting the number of seating arrangements one can create for a set number of people. For example, using only factorial, one can now answer the question on everyone's mind;

How many ways can Mrs. Bailey rearrange our class if she only uses the first 10 chairs?

______	______	______	______	______	______	______	______	______	______
1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th

For the first student, there are 32 chairs from which to choose, for the second student, there are only 31 chairs from which to choose, etc, etc..

This is called a ***10-permutation***.

$$P\begin{pmatrix}32\\10\end{pmatrix} \quad \text{or} \quad P(32,10) \quad \text{or} \quad {}_{32}P_{10}$$

Furthermore, the number of rearrangements of the 10 students in the 32 chairs can also be given by

$$\frac{32!}{32-10!}$$

<u>Combination:</u> An *r*-combination is the number of ways to choose *r* students if order doesn't matter.

and the definition of

$$\frac{n!}{(n-r)!r!} = C\begin{pmatrix}n\\r\end{pmatrix}$$

□

Example 3: Suppose the band of 97 students no longer wanted to award first, second and third prize, but instead, wanted to honor to top three students in band. How many different ways could 3 students be chosen from a band of 97 students?

Practice A: How many different hands of 5 cards are possible from a deck of 52 cards?

BASIC COUNTING TECHNIQUES

Counting Lists of Numbers

Usually, the first part of problem requires to find out how many numbers there are in a list. Never guess this information. Instead, ask yourself easier questions, then figure out a formula.

For example, suppose the question you are requires you to answer

How many multiples of 3 are there between 5 and 211, inclusive?

First, ask yourself:

1) How many numbers are there between 5 and 22, inclusive?

Did you say "17"?
I hope not. If I shift everything to the left by 4, I get the numbers between 1 and 18, inclusive. Since I didn't subtract anything, the quantity of numbers should stay the same. So the answer is ________.

2) How many multiples of 3 are there between 16 and 32. inclusive?
First, divide 5 and 32 by 3? This makes our list,

$$\frac{16}{3}, 6, \ldots, 10, \frac{32}{3}$$

Take away the non-integral numbers, then shift down by 5, and it yields

$$1, \ldots, 5$$

Verify this method works by listing the numbers between 5 and 22 and circling the multiples of 3.

$$16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32$$

3) Now use this method to answer the original question.

How many multiples of 3are there between 5 and 211, inclusive?

Exercises:

1) How many multiples of 7 are less than 150?

2) How many perfect squares are thre between 5 and 211?

Combinations and Permutations

Independent Events

When counting the number of outcomes in a series of events that do not affect each other, we use multiplication.

For example, if you were asked *How many outfits can one make with 4 pairs of pants and 3 shirts?*

One would answer 12 outfits, because the shirt chosen does not affect the choice of pair of pants.

<u>Arrangements</u>

You may be familiar with the concept of factorial as a way to count the number of ways to rearrange a set number of objects. For example, if you were asked *How many ways can one rearrange n books on a shelf?* One would answer, *n!*.

$$n! = n\cdot(n-1)\cdot(n-2)\cdot(n-3)\ \ldots\ 3\cdot 2\cdot 1$$

Combinatorically, we define n factorial to be the number of ways to rearrange n objects.

<u>Exercise:</u> How many ways are there to rearrange a deck of 52 cards?

<u>Permutations</u>

A permutation is a formula used to count how many ways one can arrange r objects from a set of n objects.

For example, if one was asked *How many ways can one rearrange 10 trophies in a row, on a shelf, from a set of 32 trophies?* One would answer

$$P\binom{32}{10} \quad \text{or} \quad P(32,10) \quad \text{or} \quad {}_{32}P_{10}$$

This is called a ***10-permutation***.

The number of rearrangements of the 10 trophies from a set of 32 can be given by

$$\frac{32!}{32-10!}$$

<u>Exercises:</u>

1) Ten women sit in 10 seats in a line. All of the 10 get up and then reseat themselves using all 10 seats. In how many ways can the women be reseated?

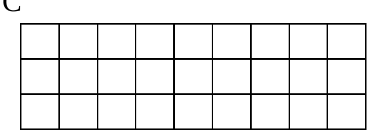
2) There are 2 Senators from each of the 50 states. We wish to make a 3-Senator committee in which no two members are from the same state. How many possible ways can we make this committee?

3) Let ABCDEFGH be a cube.

a) How many different line segments can be formed by connecting the vertices of the cube?

b) How many different paths are there from C to D on the grid shown below if you are only allwed to go down or right?

C



D

Combination

An *r*-combination is the number of ways to choose *r* students if order doesn't matter. All that needs to be done is to divide the permutation by *r!*, in order to lose the number of ways to rearrange the *r* objects chosen.

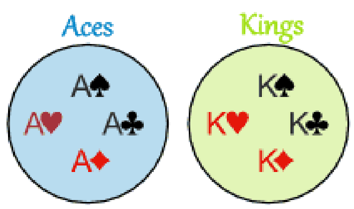
$$\frac{n!}{(n-r)!r!} = C\binom{n}{r}$$

Exercise: *How many ways are there to get a full house in a deck of 52 cards?*

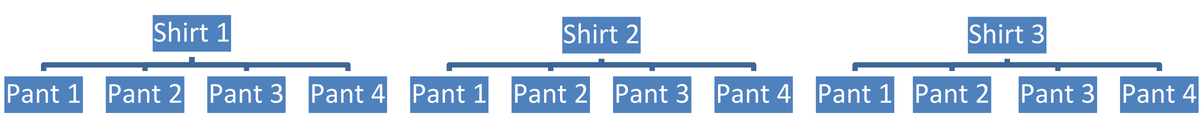
Casework:

Sometimes it is necessary to break up a problem into two or more exclusive cases. Do **NOT** confuse this with independent cases.

- Exclusive means that the set of situations do not intersect, and are therefore disjoint.
  Example: How many ways are there to choose an Ace or King from a deck of 52 cards, of only one card is chosen at random? - ADD



- Independent means that the events may occur simultaneously, but do not affect the outcome of each others occurrence.
  Example: Choosing Outfits - MULTIPLY



Exercise: How many pairs of positive integers (*m*, *n*) satisfy $m^2 + n < 22$?

* Be very careful to organize your cases so that you don't count an event more than once.

Complementary Counting:
This is the technique in which we count what we don't want versus what we do want.

For example, suppose you were confronted with the problem:

The Smith family has 4 boys and 3girls. In how many ways can they be seated in a row of 7 chairs so that at least 2 boys are next to each other?

Rather than figure this out directly, we can ask ourselves, "how many ways can they be seated so not(at least 2 boys are next to each other?". Then it becomes are problem to find an equivalent statement to:

not(at least 2 boys are next to each other)
not (there are two boys sitting next to each other)
all boys are not sitting next to each other

Lucky for us, there is only one way that the familiy can sit next to each other so that all boys sith apart from each other; BGBGBGB. There are _______ways to sit the boys in this position, and ______ ways for the girls to sit in this position. So there are a total of ____________ ways for this seating position to take place. The total number of seating arrangements is _________. Therefore, there are ____________ - ___________ number of ways for the family to be seated with the original constraints.

Exercises:

1) How many 4-letter words with at least one vowel can be constructed from the letters A, B, C, D, and E?

2) How many 5-digit numbers have exactly one zero?

General Hints:

- If many restrictions are placed on the situations we need to count, it is often better to deal with the harshest restrictions first.
- When faced with independent choices, we mutliply.
- When faced with exclusive choices, we add.
- Always organize cases to avoid double counting.
- When cases become too complicated, it may be best to use complimentary counting.
- Sometimes it helps to look at the simplest cases of a more general problem to find a pattern.

Challenge Problems:

1. Let S be a cube. Compute the number of planes which pass through at least three vertices of S.

2. How many zeroes do we write when we write al lthe integers from 1 to 256 in binary?

3. How many 5-letter strings of letters have at least two consecutive letters which are the same?

Overcounting

When counting distinct arrangements with n - indistinguishable items, you divide by $n!$

For example, suppose you were asked to find the number of distinct arrangements of the word TATTER.

There are 6 total letters, which would yield ______ total arrangements, but there are 3 T's, which means we over counted. Therefore, we divide by 3!, giving us the answer of _______.

Counting with symmetries

When confronted with a situation which involves symmetries, we first ignore the symmetry, then divide to account for the symmetries.

For example, suppose you were asked to find the number of different ways 6 people can be seated at a round table? The number of ways that one can rearrange 6 people is 6!. However, there are 6 possible rotations of the same arrangement. Therefore we must divide by 6, yielding 5!.

Exercise: In how many distinct ways can 4 keys be placed on a keychain? Two arrangements are not considered different if the keys are in the same order?

Challenge Problems:

1) How many diagonals of a regular octagon are not parallel to one of the sides?

2) A regular tetrahedron is a triangular pyramid whose faces are all equilateral triangles. How many distinguishable ways can we paint the four face of a regular tetrahedron with red, blue, green, and orange paint such that no two faces are the same color?

3) In how many distinguishable ways can 8 people sit a square table, 2 people to a side?

4) Twenty married couples are at a party. Every man shakes hand with everyone except himself and his spouse. Half of the women refuse to shake hands with any other women. The other 10 women all shake hands with each other, but not with themselves. How many handshakes are there at the party?

5) An integer is called *snakelike* if its decimal representation $a_1a_2a_3\cdots a_k$ satisfies $a_i < a_{i+1}$ if i is odd, and $a_i > a_{i+1}$ if i is even. How many snakelike integers between 1000 and 9999 have four distinct digits?

Hints

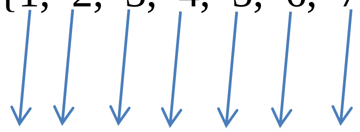
- When stuck on a problem that has variables, plug in some numbers, solve and look for a pattern.
- When you are unsure how to solve a complicated problem, look at a simpler version of the same problem and look for patterns
- Be flexible with approaches. Don't get stuck on one approach.

<u>One to one correspondence!</u>

A one-to-one correspondence is a way of aligning the elements of two sets so that they are each accounted for by their counterpart exactly once. In mathematics, we call this an invertible function.

Example:

{1, 2, 3, 4, 5, 6, 7, …..}



{2, 4, 6, 8, 10, 12, 14, …….}

As you can see, there is a one to one correspondence between even natural numbers and all the natural numbers.

<u>Theorem:</u> Two sets have the same number of elements if there is a one-to-one correspondence between them.

<u>Example:</u> A dog trainer wants to buy 8 dogs, all of which are either Cocker Spaniels, Irish setters, or Russian wolfhounds. In how many ways can they make the choice?

<u>Solution:</u> Instead of trying to count each sequence, or trying to make cases, let us try creating a one-to-one correspondence. We begin by creating a table and computing some examples to see how we can make a one-to-one correspondence

Cocker Spaniels	Irish Setters	Russian Wolfhounds
DDD	DD	DDD
DD	DDD	DDD
DDDD	DD	DD

I can reduce this to information

DDD|DD|DDD, DD|DDD|DDD, DDDD|DD|DD

Note, the placement of the bars tells me how many of each do there are. I can make this clearer by replacing the bars with the letter B

DDDBDDBDDD, DDBDDDBDDD, DDDDBDDBDD

Now I see that the number of ways to choose these dogs is in one-to-one correspondence between the number of ways to rearrange 8D's and 2'Bs in a row. Since it wasn't specified that a dog of each kind had to be chosen, it is clear that the placement of the B's has no restrictions. This means the answer is

$$C\binom{10}{2} = \frac{10 \cdot 9}{2!} = 45$$

Exercise 1: Find a general theorem for the number of ways to "buy" *n* dogs if there are *r* varieties to choose from.

Exercise 2: Prove that there is a one to one correspondence between the number of ways to buy *n* dogs if there *r* varieties to choose from, and the number of solutions to the equation

$$x_1 + x_2 + x_3 + \cdots + x_r = n$$

For simplicities sake, we will denote $C\begin{pmatrix} n+r-1 \\ r-1 \end{pmatrix}$ by $\begin{pmatrix} n+r-1 \\ r-1 \end{pmatrix}$.

Exercise 3: Use the information above to find the number of non-negative solutions to $x_1 + x_2 + x_3 \leq 50$.

Generalized Binomial Theorem

$(x+y)^n = \sum_{k=0}^{\infty} C\binom{n}{k} x^k y^{n-k}$ where $C\binom{n}{k} = \frac{(n)(n-1)(n-2)(n-3)\cdots(n-k+1)}{k!}$ and *n* is a real number.

The proof of this theorem lies in the Taylor Series Expansion of $(x+y)^n$, or one can prove this by induction.

Example: Compute the coefficient of x^6 in $(1-x)^6$

Example: Compute the coefficient of x^{10} in $(1-x)^{-15}$

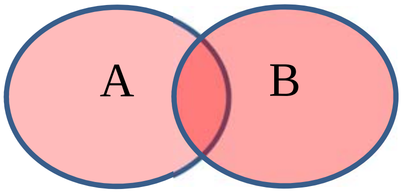
Set Theory

Sometime it helps to count by looking at the objects as living in a set. As a reminder, here are some basic set theoretic definitions.

set: A set is a collection, S, with a well-defined property.

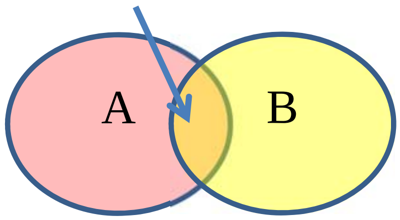
union: Let *A* and *B* be sets. The *union* of *A* and *B* is defined to be the set

$$A \cup B = \{x \in A \text{ or } x \in B\}$$



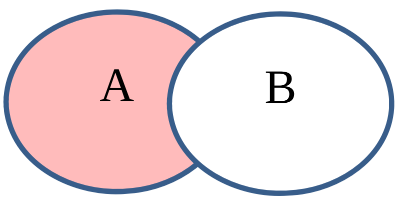
intersection: Let *A* and *B* be sets. The *intersection* of *A* and *B* is defined to be the set

$$A \cap B = \{x \in A \text{ and } x \in B\}$$



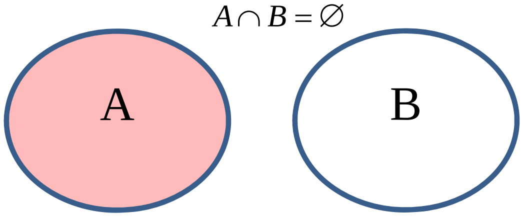
set minus: Let *A* and *B* be sets. We define *A minus B* is defined to be the set

$$A \setminus B = \{x \in A \text{ and } x \notin B\}$$



disjoint: Let *A* and *B* be sets. We say that *A* and *B* are *disjoint* if and only if

$$A \cap B = \varnothing$$



cardinality: The cardinality of a set *A*, $|A|$, is the number of elements in *A*.

Cardinality of union**:**

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Exercise 1: There are 100 students taking language classes at Austin High School. If 60 are taking German and 75 are taking Spanish and these are the only languages taught, how many students take both Spanish and German.

Exercise 2: How many positive integers less than or equal to 1000 do not have 2, 3 or 5 among their prime factors?

Generating Functions

A **generating function** is a power series, $\sum_{k=0}^{\infty} A(k)x^k$, whose coefficients, $A(k)$, gives information about the *k*. Usually this information is regarding the number of ways *k* can occur, or the probability of *k* occuring.

Example: How many ways can we collect $*d* from 5 people if they are only allowed to give $0, $1, $3.

Solution: Let *A*(*k*) be the number of ways we can collect $*k* from each of the 5 people if they are only allowed to give $0, $1, $3. Then

k	$A(k)$
0	1
1	1
2	0
3	1
4	0
5	0
n > 5	0

the generating function for each person is $(1+x+x^3)$. Therefore, the generating function for 5 people is

$$(1+x+x^3)^5$$

and the number of ways we can collect $*d* is the coefficient of the *d* th term.

Let us test this. Suppose we ask the question.
How many ways are there to collect \$6 from 5 people if each person can give \$0, \$1, or \$3.
Then we can count this out by hand;

0 + 0 + 0 + 3 + 3 → $\frac{5!}{3!2!} = 10$

0 + 1 + 1 + 1 + 3→ $\frac{5!}{3!} = 20$

This is the only way to make \$6 with 5 \$3's, \$0's and \$1's, so the answer should be 30.

Now let us try it the other way;

$$(1+x+x^3)^5 = C\binom{5}{0}(1+x)^0(x^3)^5 + C\binom{5}{1}(1+x)^1(x^3)^4 0 +$$
$$C\binom{5}{2}(1+x)^2(x^3)^3 + C\binom{5}{3}(1+x)^3(x^3)^2 +$$
$$C\binom{5}{1}(1+x)^4(x^3)^1 + C\binom{5}{0}(1+x)^5(x^3)^0$$

We can see that there is no way to get the x^6 term from the first 3 products or the last one, so we will disect the fourth and fifth term

$$C\binom{5}{3}(1+x)^3(x^3)^2 = 10(1+3x+3x^2+x^3)(x^6) = 10x^6+30x^7+30x^8+10x^9$$

$$C\binom{5}{1}(1+x)^4(x^3)^1 = 5(1+4x+6x^2+4x^3+x^4)(x^3) = 5x^3+20x^4+30x^5+20x^6+5x^7$$

From this we can see that the coefficient of x^6 will be 30.

Now, you may say, "Mrs. Bailey, that was much more complicated than what you did initially?"

To which I would respond, " Yes, but I chose an example I knew I could compute the old way quickly. What if we were given something more complicated that would be much more inefficient to do the old way?"

"Like what?" You ask.

"Well, I'm glad you asked that. Let me show you."

Please excuse Mrs. Bailey. It is late, and I think she is currently out of service. She will return later when she is no longer insane.

Mrs. Bailey promises not to have imaginary conversations with you guys anymore during this lesson.

Exercise 1: Ten people with one dollar each and one person with three dollars get together to buy an eight-dollar pizza. In how many ways can they do it?

Exercise 2: In how many ways can we get a sum of 25 when 10 distinct 6 sided fair dice are rolled?

[Hint: $x + x^2 + x^3 + x^4 + x^5 + x^6 = x(1 + x + x^2 + x^3 + x^4 + x^5) = x(\frac{1-x^6}{1-x})$]

Definition of Probability:

Probability of an event E occuring in a sample space S is $P(E)=\frac{|E|}{|S|}$,

where | E | is the number of ways E can happen, and |S| is the cardinality of the sample space, S.

Most of the basics of probability involve different counting techniques. The only difference is that now you are dividing that number by the cardinality of the sample space.

Exercises:

1) Ten people are sitting around a table. Three of them are chosen at random to give a presentation. What is the probability that the three chosen at random are sitting in three consecutive seats?

2) Two different two digit natural numbers are randomly chosen and multiplied together. What is the probability that the product is even?

3) What is the probability of getting a full house?

Probability of A or B:

If A and B are two distinct events in a sample space S, then the probabilty of **A or B** occuring is

$$P(A \text{ or } B)=P(A)+P(B)-P(A \text{ and } B)$$

Do you see a similarity here in the counting of elements in the union of two sets, that because it is exactly the same thing. We can rewrite the equation above as

$$\frac{|A\cup B|}{|S|}=\frac{|A|}{|S|}+\frac{|B|}{|S|}-\frac{|A\cap B|}{|S|}$$

Clearly, if A and B are **mutually exclusive**, or the sets A and B are disjoint, then we get

$$P(A \text{ or } B)=P(A)+P(B)$$

Exercises:

1) A fair coin is flipped 7 times. What is the probability that at least 5 of the flips come up heads?

2) A card is chosen at random from a standard deck of 52 cards. What is the probability the card is a Queen or a Diamond?

3) A bag has 3 red marbles an *k* white marbles, where *k* is a positive integer. Two of the marbles are chosen at random from the bag. If the probability that the two marbles are the same is $\frac{1}{2}$, then what is *k*?

Probability of Not E:

Let E be the set of possible desired events, and S be the set of possible outcomes in the sample space.

Then $S \setminus E$ = the set of possible events in the sample space that are not desired.

Then $P(\neg E) = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - P(E)$ is the probability that NOT E will occur

Probability of Independent Events:

If A and B are possible outcomes for two independent events, then

$$P(A \text{ and } B) = P(A)P(B)$$

Probability of Dependent Events:

Make the events be independent and multiply.

Example: Two cards are dealt from a standard deck of 52 cards. What is the probability that the first card dealt is a diamond and the second card dealt is a spade?

Solution: There are 13 diamonds and 13 spades. So the probability of the first card being a diamond is _________. Once the first card is picked, there are _____ cards left in the deck. The picking of the second card is now independent of the first card picked. The probability that the second card is a spade given the first one is a diamond is ________. Therefore the probability that you pick, first, a diamond, then a spade is ____________.

Exercise: A bag has 4 red and 6 blue marbles. A marble is selected and not replaced,t hen a second is selected. What is the probability that both are the same color?

Shooting Stars:

Example: Becky lives in a place with a very clear sky. On a Friday night, ther eis a 60% chance that she will see a shooting star in any given hour. If Becky goes outside and watches the sky for two hours, what is the probability that she'll se a shooting star?

Solution: Break the desired event into mutually exclusive events (cases)

Case 1: Becky sees a shooting star in the first hour, but not the second.

Case 2: Becky sees a shooting star in the second hour, but not the first.

Case 3: Becky sees a shooting star in both hours.

Exercise: On Saturday night, Becky again goes stargazing. This time, conditions are better, and there's an 80% chance that she will see a shooting star in any given hour. We assume that the probability of seeing a shooting star is uniform the entire hour. What is the probability that Becky will see a shooting star in the first 15 minutes?

(Hint: Let p be the probability that she sees a shooting start in a 15 minute period of time. Then find the probability that she will not see a shooting star in the entire hour as a experssion in p).

Geometric Probability:

The geometric probality of a success ful even occuring is

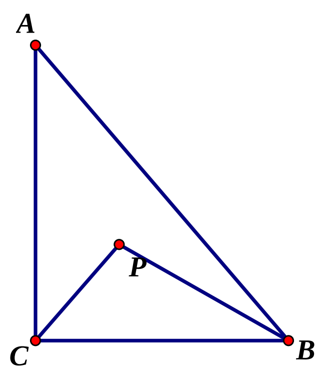
$$P(\text{ success }) = \frac{\text{size of successful region}}{\text{size of possible region}}$$

Exercises:

1) Three points, *x*, *y*, and *z* are chosen at random on the unit interval (0,1). What is the probability that $x \le y \le z$.

2) What is the probability that if three points are chosen at random on the circumference of a circle, then the triangles formed by connecting the three points does not have a side with length greater than the radius of the circle?

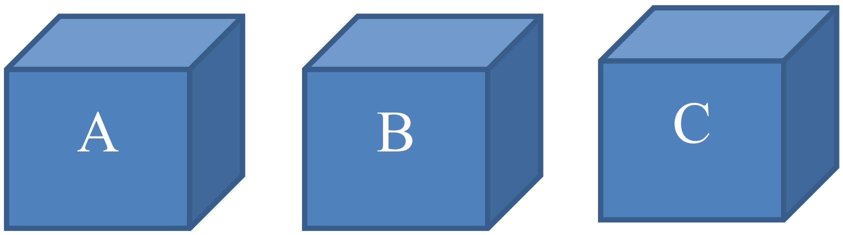
3) A point, *P*, is randomly chosen in the interior of the right triangle *ABC*, as shown below. What is the probability that area of *PBC* is less than half of the area of *ABC?*



Conditional Probability

Let's Make a Deal!!

Suppose I have three boxes.
In one box is $1,000,000. In the other two there is nothing.
Suppose you choose box C.
What is the probability that you chose correctly? ______
What is the probability that you chose wrongly? ______



Now let us suppose that I open box, B, and its empty. I ask you is you want to switch your choice.

Here is the thought process going on in the brillian head of yours:

What is the probability that C is the wrong box? Not 1/2, but still 2/3.
Since we now know that B is not the right box, the probability that C is the wrong box and the probability that A is the wrong box must add up to 1. Therefore, the probability that A is the wrong box is 1/3. Which means the probability that A is the right box is 2/3. SWITCH TO A!!!!!

Good job! This problem is called the Monty Hall problem. A statistician named Steve Selvin in the *American Statistician* in 1975 (Selvin 1975a). The problem was answered by Marilyn vos Savant in a column in *Parade* magazine in 1990. Her answer upset so many mathematician, that Parade magazine received 10,000 letters, 1,000 from PhD's in math, exclaiming how wrong she was. Even Paul Erdos, an extraordinarily renowned mathematician was utterly unconvinced she was wrong, until a computer simulation performed 10,000 trials and supported her claim. If you are interested in trying a simulation, go to the websites

http://math.ucsd.edu/~crypto/cgi-bin/MontyKnows/monty2?0+4338

http://www.nytimes.com/2008/04/08/science/08monty.html?_r=0

The problem above is an example of conditional probability. The conditional probability of an event **B** given an event **A**, is the probability that **B** will occur given the knowledge that event **A** has already occurred. This probability is written $P(B|A)$. There are two cases in conditional probability

- **B** and **A** are independent events: $P(B|A) = P(B)$
- **B** and **A** are dependent events: $P(B|A) = \dfrac{P(B \text{ and } A)}{P(A)} = \dfrac{P(B \cap A)}{P(A)}$

<u>Example:</u> Suppose I have two cards. Card A has a blue side and a red side. Card B has two red sides. I choose one at random and place it on the table. The top is red. What is the probability that the other side is red?

Solution: If you said 1/2, read the first page again. Go on. I'll wait.
What is **B**?
B = {the possibile outcomes in which both sides are red}. **A** = { a given side is red}
$P(B \cap A) = \frac{|B \cap A|}{|S|} = \frac{1}{2}$ and the $P(A) = \frac{|A|}{|S|} = \frac{3}{4}$. Therefore, the probability of **B** given **A** is

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}.$$

<u>Exercise:</u> Bag X has 5 white marbles and 2 black marbles. Bag Y has 3 white marbles and 5 black marbles. A bag is chosen at random and amarble taken from the bag. The marble is white. What is the probability that the bag was bag X?

Solution: A = { events that the ball is white }, B = { events that Bag X is chosen }

Step 1: Compute $P(B \cap A)$.

Step 2: Compute $P(B) = P(\text{ white marble from bag X}) + P(\text{ white marble from bag Y})$

Step 3: Compute $P(B|A) = \frac{P(B \cap A)}{P(A)} =$

Problems:

1) In the World Series, twoteams play each other repeatedly until one team has won a total of 4 games, then the series ends. If each team is equally likely to win each game, what is the probability that the series ends in exactly 6 games?

2) If 3 successive rolls of a die are all greater than three, what is the probability that they are all the same?

3) If a number is selected at random from the set of all five digit natural numbers in which the sum of the digits equal to 43, what is the probability that this number will be divisibly by 11?

4) A point P is chosen at random in the coordinate plane. What is the probability that the unit circle with center P contains exactly two points with integer coordinates (lattice points).

5) Three points A, B, and C are selected at random on the circumference of a circle. Find the probability that the points lie on a semicircle.

6) Three number are chosen at random between 0 and 1. What is the probability that the difference between the greatest and the least is less than $\frac{1}{3}$?

Name:

AIME: Probability

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Problem 1. Starting at $(0, 0)$, an object moves in the coordinate plane via a sequence of steps, each of length one. Each step is either left, right, up or down, and all four are equally likely. Let p be the probability that the object reaches $(2, 2)$ in six or fewer steps. Given that p can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

Problem 2. A fair die is rolled four times. The probability that each of the final three rolls is at least as large as the roll preceding it may be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 3. Three number a_1, a_2, a_3 are drawn randomly and without replacement from the set $\{1, 2, 3, \ldots, 1000\}$. Three other numbers $\{b_1, b_2, b_3\}$ are then drawn randomly and without replacement from the remaining set of 997 numbers. Let p be the probability that, after suitable rotation, a brick of dimensions $a_1 \times a_2 \times a_3$ can be enclosed in a box of dimension $b_1 \times b_2 \times b_3$ with sides of the brick parallel to the sides of the box. If p is written as a fraction in lowest terms, what is the sum of the numerator and denominator?

Problem 4. Forty teams play a tournament in which every team plays every other team exactly once. No ties occur, and each team has a 50% chance of winning any game it plays. The probability that no two teams win the same number of games in $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $\log_2(n)$.

Problem 5. Let S be the set of points whose coordinates x, y and z are integers that satisfy $0 \leq x \leq 2$, and $0 \leq y \leq 3$, and $0 \leq z \leq 4$. Two distinct points are randomly chosen from S. The probability that the midpoint of the segment they determine also belongs to S is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 6. When a certain biased coin is flipped 5 times, the probability of getting heads exactly once is not equal to zero and is the same as that of getting heads exactly twice. Let $\frac{i}{j}$, in lowest terms, be the probability that the coin comes up heads exactly 3 times outs of 5. Find $i + j$.

Problem 7. Six men and some number of women stand in a line in random order. Let p be the probability that a group of at least four men stand together in a line, given that every man stands next to at least one other man. Find the least number of women in the line such that p does not exceed 1 percent.

Problem 8. A bug starts at a vertex of an octahedron. On each move, it randomly selects one of the four vertices adjacent to the one where it is currently located, and crawls along an edge of the octahedron to that vertex. Given that the probability that the bug moves to its starting vertex on its seventh move is $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.

Problem 9. Let S be a set with six elements. Let P be the set of all subsets of S. Subsets A and B of S, not necessarily distinct, are chosen independently and at random from P. The probability that B is contained in at least one of A or $S - A$ is $\frac{m}{n}$, where m, n, and r are positive integers, n is prime, and m and n are relatively prime. Find $m + n + r$.

Problem 10. The numbers $1, 2, 3, 4, 5, 6, 7$ and 8 are randomly written on the faces of a regular octahedron so that each face contains a different number. The probability that now two consecutive numbers, where 8 and 1 are considered to be consecutive, are written on faces that share an edge is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

1 Answers

(1) 67

(2) 79

(3) 5

(4) 742

(5) 200

(6) 283

(7) 594

(8) 149

(9) 710

(10) 85