Polynomial Division:

We already learned the how to divide polynomials the long way; by using the division algorithm. Now we will see how to divide polynomials by a linear factor with a leading coefficient of 1 (monic).

Synthetic Division: Only works when the divisor is of the form "x-a"

Divisor

$$x-3\overline{\big)\,x^3-3x^2+4x-2}$$

divisor Dividend

Step 1: Write Coefficients of the dividend inside the box

Step 2: Write the negative of the constant coefficient of the

Step 3: Make sure you write all coefficients (including 0's!!!)

Step 4: Drop, multiply, add

$$3\,\big|\;1 \quad -3 \quad 4 \quad -2$$

Fundamental Theorem of Algebra:

The fundamental theorem of algebra states that if $p(x)=a_0+a_1x+a_2x^2+\cdots a_nx^n$ is a nonconstant polynomial (deg (p) > 1) with real or complex coefficients, then it must have at least one complex root.

The proof of this theorem requires some Complex Analysis (Calculus on Complex Functions) and is a consequence of Liouville's Theorem or the Maximum Modulus Principle. Therefore, we will not be proving it today.

An extremely important corrolary of the Fundamental theorem of algebra is that if $p(x)=a_0+a_1x+a_2x^2+\cdots a_nx^n$ is a n-degree polynomial with real or complex coefficients, then it must have n complex roots. Why?

Remainder Theorem: If you divide a polynomial, $f(x)$, by $x-a$, it will have remainder $f(a)$.

Proof: Use the division algorithm on the polynomial and you will get

$$f(x)=(x-a)q(x)+r(x) \text{ where } \deg(r(x))<\deg(x-a)$$

However, since deg(x - a) = 1, then deg(r(x)) = 0, which means it's a constant.
Let us call $r(x) = r$.
Therefore,

$$f(a)=(a-a)q(a)+r$$
$$f(a)=r$$

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Factor Theorem: A corollary of the remainder theorem, this theorem states that x - a is a factor of $f(x)$ if and only if $f(a) = 0$, a is a root of $f(x)$.

Proof:

Rational Roots Theorem:

Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$, where $a_i\in\mathbb{Z}$. Then $\frac{p}{q}$ is a possible rational root of $f(x)$ if p is a factor of a_0 and q is a factor of a_n

Proof: Suppose $\frac{p}{q}$ is a root of $f(x)$. Assume that p and q are relatively prime.
Then

$$f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$$
$$0=a_0+a_1(\frac{p}{q})+a_2(\frac{p}{q})^2+\cdots+a_n(\frac{p}{q})^n$$
$$0=q^na_0+q^{n-1}a_1p+q^{n-2}a_2p^2+\cdots+a^np^n$$

by clearing the denominator. To see that p is a factor of a_0, we need only show that $a_0=0 \bmod p$, and similarly to show that q is a factor of a_n.
If we mod out the last equation by p, we get $0 \bmod p=q^na_0$. Since q is relatively prime to p, then $0 \bmod p=a_0$. A similar argument can be made to show that $0 \bmod q=a_n$.

Descartes Rules of Signs

I never paid much attention to this theorem, but it may help you guys. The Descartes Rules of Signs says that the number of sign changes in the coefficients of $f(x)$ tells you the maximum number of positive real roots $f(x)$ has. The number of sign changes in the coeffecients of $f(-x)$ tells you the maximum number of negative real roots that $f(x)$ has..

The proof of this theorem is not hard, but it takes about 10 Lemmas, 14 Corrollaries and a Partridge in a pear tree. I will hyperlink to it on my website if anyone is interested.

Another useful tool in helping us pinpoint the real roots of a polynomial is finding upper and lower bounds for the roots.

Upper and Lower Bounds for the Roots

Suppose $f(x)$ is a polynomial with real coefficients. Further suppose for some real number $a > 0$, $f(x) = (x-a)q(x) + r$ where $r > 0$ and $q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$, where $b_i > 0$. Then for any real number $c > a$, we get $f(c) = (c-a)(b_0 + b_1 c + b_2 c^2 + \cdots + b_n c^n) + r > 0$.This means there can be no real roots to f greater than a, which makes a an upper bound for the real roots of f.

A similar argument can be made in the negative direction . Just change all the signs above.

Coefficients and Roots

Suppose f is a monic polynomial (with leading coefficient 1) of degree n with real coefficients. By the Fundamental Theorem of Algebra, it has n complex roots. Let us look at f in terms of its coeffecients and it's roots simoultaneously.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + x^n$$
$$f(x) = (x - r_1)(x - r_2)(x - r_3)\cdots(x - r_n)$$

If we mutliply the roots out you will see that

$$f(x) = x^n - (r_1 + r_2 + r_3 + \cdots + r_n)x^{n-1} + (\sum_{i \neq j} r_i r_j)x^{n-2} - (\sum_{i \neq j \neq k} r_i r_j r_k)x^3 + \cdots + (-1)^n \prod_{i=1}^{n} r_i$$

The constant coefficient gives us the product of all the roots. The coefficient of n -1 gives us the sum of all the roots, an so forth. As you can see, there are signs to consider, but they are easily computed by taking -1to the power of the number of roots you are multiplying together.

If the polynomial is not monic, then just divide all the coefficients by the leading coefficient, and now it is monic. The rootsof f do not change if you divide f by a nonzero real number, so no harm done. With a monic polynomial, you can revert to the previous case.

Problems:

1) The equations with roots $3+\sqrt{2}, 3-\sqrt{2}, 3+i\sqrt{2}, -3-i\sqrt{2}$ is in the form $x^4+ax^3+bx^2+cx+d=0$, find *a*+*b*+*c*+*d*,

2) A polynomial *p* contains only terms of odd degree. When *p* is divided by *x* - 3, the remainder is 6. What is the remainder when *p* is divided by x^2-9.

3) Suppose $p(x)=a_0+a_1x+a_2x^2+\cdots a_nx^n$ is a polynomial with integer coefficients. Show if $p(0)$ and $p(1)$ are both odd, then *p* has no integer roots.

4) If $x^4+4x^3+6px^2+4qx+r$ is divisible by x^3+3x^2+9x+3 then find $(p+q)r$.

5) Solve the $(x+1)(x+2)(x+3)(x+4)=-1$.

6) Give the remainder when $x^{203}-1$ is divided by x^4-1.