Series Convergence Tests: Part 4

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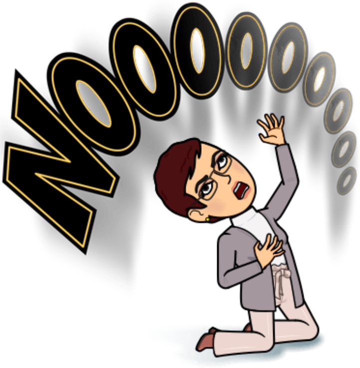
We have 3 more tests to go

Today we will cover 2 of the last 3.
The Ratio Test and The Root Test. (p. 627 in your book)



Ratio Test

Now consider a sequence with positive terms, a_n such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$, then

$$\frac{a_{N+1}}{a_N} \approx r$$

for large enough N.
Which means that for large enough N,

$$a_{N+1} \approx r a_N$$

In other words, the sequence starts behaving like a geometric series when N is large enough.

For $k \geq N$

Therefore, for $k \geq N$, we know that

$$\begin{aligned} a_k + a_{k+1} + a_{k+2} + \cdots &\approx a_k + ra_k + r^2 a_k + r^3 a_k + \cdots \\ &= a_k(1 + r + r^2 + r^3 + \cdots) \\ &= a_k\left(\frac{1}{1-r}\right) \end{aligned}$$

So the sum from N to ∞ converges, and the sum from 1 to $N - 1$ is a finite sum, so it must converge.
This means that the series converges if and only if $r < 1$.

The Ratio Test

If $\sum_{n=1}^{\infty} a_n$ is an infinite series with positive terms and $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$, then

- $L > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges.
- $L < 1$, then $\sum_{n=0}^{\infty} a_n$ converges.
- $L = 1$ then more information is needed.

Example 1

$$\sum_{k=1}^{\infty} \frac{10^k}{k!}$$

If you forgot, $k! = 1(2)(3)(4)\cdots(k-1)(k)$
Since the series has positive terms let's see what the ratio test says:

Example 1

$$
\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}} \\
&= \lim_{n\to\infty} \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} \\
&= \lim_{n\to\infty} \frac{10}{(1)(2)(3)\cdots(n)(n+1)} \cdot (1)(2)\cdots(n) \\
&= \lim_{n\to\infty} \frac{10}{n+1} \\
&= 0 \\
&< 1
\end{aligned}
$$

Therefore, by the ratio test, the series converges.

Root Test

Not consider a series with positive terms, a_n, such that $\lim_{n\to\infty}\sqrt[n]{a_n} = r$, then

$$\lim_{n\to\infty}\sqrt[n]{a_n} = r$$

$$\lim_{n\to\infty} a_n = r^n$$

$$a_n \approx r^n \text{ , if } n \text{ is large enough}$$

So we can see a pattern of approximation as we continue through the series.

Which essentially means that as n gets large, $a_{n+1} \approx r^{n+1}$. Therefore, the tail end of the series

$$a_k + a_{k+1} + a_{k+2} + \cdots \approx r^k + r^{k+1} + r^{k+2} \cdots$$

$$= r^k(1 + r + r^2 + r^3 + \cdots)$$

$$= r^k\left(\frac{1}{1-r}\right) \text{ if and only if } r < 1.$$

Which leads us to the root test.

The Root Test

If $\sum_{n=1}^{\infty} a_n$ be an infinite series with nonnegative terms and $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, then

- $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges
- $0 < L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges
- $L = 1$ more information needed

Example 2

$$\sum_{k=1}^{\infty} \frac{10^k}{k^7}$$

Example 2

Since the series has positive terms

$$\sum_{k=1}^{\infty} \frac{10^k}{k^7}$$

we can use the root test

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \sqrt[n]{\frac{10^n}{n^7}} \\ &= \lim_{n\to\infty} \frac{10}{\sqrt[n]{n^7}} \\ &= 10 \lim_{n\to\infty} \frac{1}{(\sqrt[n]{n})^7} \\ &= 10(1) \\ &> 1 \end{aligned}$$

Therefore, by the root test, the series diverges.

How do you know which one to use?
A series flow chart is available and linked to this assignment.
Use it well.