Consider the differential equation

$$y''+k^2y=0$$

has particular solutions $y_1=\sin(kx)$ and $y_2=\cos(kx)$.

In general, any ***linear combination*** of y_1 and y_2, $c_1y_1+c_2y_2$ where $c_1,c_2\in\mathbb{R}$ is also a solution to the equation above. The reason for this lies in Linear Algebra. Through our study of Linear Algebra, we will learn that the set of differentiable functions from $\mathbb{R}^n\to\mathbb{R}^m$ is a ***vector space***, which we will define below.

Vector Space: A vector space is a set, V, along with two operations called ***vector addition***, +, and ***scalar multiplication*** with the following properties

(1.1)[Zero Vector] There exists $0\in V$ such that for all $v\in V, v+0=v$.

(1.2)[Additive Inverse Vector] For all $v\in V$, there exists $-v\in V$ such that $v+-v=0$

(1.3) [Associativity] If $v,w,u\in V$, then $(v+w)+u=v+(w+u)$.

(1.4) [Commutativity] For all $v,w\in V$, $v+w=w+v$

(1.5) [Vector Addition Closure] For all $v,w\in V$, $v+w\in V$

(1.5) [Scalar Closure] If $c\in\mathbb{R}$, then $cv\in V$

(1.6) [Identity Scalar] $1v=v$ for all $v\in V$

(1.7) [Scalar Associativity] $(c_1c_2)v=c_1(c_2v)$ for all $c_1,c_2\in\mathbb{R}$, $v\in V$

(1.8) [Distributivity] $c(v+w)=cv+cw$ and $(v+w)c=vc+wc$

Problems: First you will need to define these spaces.

1) Show the set of all continuous functions, $C_0(\mathbb{R}^n)$, from $\mathbb{R}^n\to\mathbb{R}^m$ is a vector space.

2) Are the set of polynomials with real coefficients, $\mathbb{R}[x]$, a vector space? Justify your answer.

3) Are the set of convergent sequences, $\mathbb{R}^\infty$, a vector space? Justify your answer.

Subspaces and Linear Combinations

Subspace: A *subspace,* W, of a vector space, V, is a subset of V which is also a vector space under the same vector addition and scalar multiplication.

Subspace criteria: To show $W \subset V$ is a subspace, it suffices to show that W satisfies the following properties.

1) The additive identity $0 \in W$

2) If c_1 and $c_2 \in \mathbb{R}, w_1$ and $w_2 \in W$, then $c_1 w_1 + c_2 w_2 \in W$. *We call this property "closure under addition and scalar multiplication".*

Example: Show that the set of all polynomials of degree less than or equal to *n*, $\mathbb{R}_n[x]$, forms a subspace of $\mathbb{R}[x]$.

Solution: The Zero polynomial is the polynomial whose constantly 0, so therefore, with all coefficients zero.

To show closure, suppose there exists two polynomials $p(x), q(x) \in \mathbb{R}_n[x]$. Then $ap(x) + bq(x)$ has real coefficients, and cannot have degree greater than *n*. Therefore, $ap(x) + bq(x) \in \mathbb{R}_n[x]$.

Problems:

1. Show that the set of all differentiable functions, $C_1(\mathbb{R}^n)$, is a subspace of the vector space of continuous functions, $C_0(\mathbb{R})$.

2. Determine which of the following subsets are subspaces of the indicated vector space:

a) $\{(1, y) \mid y \in \mathbb{R}\}$ in $\mathbb{R}^2$.

b) $\{(x, y) \mid x, y \in \mathbb{R} \setminus \{0\}$ and $\frac{x}{y} = 2\} \cup (0,0)$in $\mathbb{R}^2$

c) $\{(x, y, z) \mid x, y, z \in \mathbb{R}$ and $x + y + z = 0\}$ in $\mathbb{R}^3$

d) $\{(x, y, z) \mid x, y, z \in \mathbb{R}$ and $x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$

Linear Dependence and Independence:

We say a subspace, *W*, is *generated* by a set of vectors $\{v_1, v_2, \ldots v_n\}$ if for all $w \in W$, there exists real numbers $a_1, a_2, \ldots, a_n$ such that $w = \sum_{k=1}^{n} a_k v_k$. In other words, if every element in *w* is a linear combination of the v_i's. We also say $\{v_1, v_2, \ldots v_n\}$ *spans* W or $Sp(v_1, v_2, \ldots v_n) = W$.

However, we also would like the smallest possible set that spans *W*. For example, consider the vectors

$$1,\ x,\ x^2,\ 2x^2+1,\ x-2$$

in the vector space $\mathbb{R}_2[x]$. It is clear that $Sp(1,\ x,\ x^2,\ 2x^2+1,\ x-2) = \mathbb{R}_2[x]$, but do we really need all of them? We know we can write any vector in $\mathbb{R}_2[x]$ as ax^2+bx+c where *a* , *b*,and *c* are real numbers. Doesn't this mean, by definition, that they can be written as a linear combination of x^2, x ,and 1? Then we really don't need the other two vectors. What we are looking for is a spanning set of ***linearly independent*** vectors. It is easier, however, to define linear dependence than to define linear independence.

Linear Dependence: We say that a set of vectors $\{v_1, v_2, \ldots v_n\}$ is *linearly dependent* if there exists $a_1, a_2, \ldots, a_n$ where not all $a_i = 0$, such that $0 = a_1v_1 + a_2v_2 + \cdots + a_nv_n$. If this does not occur, then we call the set of vectors *linearly independent*.

Why is this the desired property that tells us that we have no extra vectors?
Well, from our example, we could see that the other two vectors could be written as linear combinations of $x^2, x,$ and 1. In particular,

$$2x^2+1 = 2(x^2)+1(1)$$
$$x-2 = 1(x)-2(1)$$

It looks a little silly, but now we see that this means

$$2x^2+1-2(x^2)-1(1) = 0$$
$$x-2-1(x)+2(1) = 0$$

In general we can do the same thing. From this point on, when we mention zero, we mean the zero vector. If a linear combination of a set of vectors equals zero, and at least one coefficient is nonzero, then we can divide by that nonzero coefficient, and isolate the corresponding vector, ***v***. This will yield a linear combination of the other vectors equal to ***v***, making ***v*** unnecessary.

To show a set is linearly independent, we need to show no such combination exists. Usually, one does this by contradiction or by contrapositive.

Example: Show that the set of vectors {(1,0,0), (0,1,0) and (0,0,1) is linearly independent.

Solution: Suppose there exists $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1) = (0,0,0)$$
$$(a_1, a_2, a_3) = (0,0,0)$$

This can only happen if $a_1 = a_2 = a_3 = 0$, which means that the set is linearly independent, by definition

Basis: A *basis* of a vector space is a linearly independent spanning set of the vector space.

There may be many possible *bases* to a given vector space, but they will always have the same number of elements. The proof of this is involved and heavy on the notation, so we will defer it to your linear algebra class. The number of vectors in a basis of a vector is called then ***dimension*** of the vector space.

Problems:

1. Show that the set {(2,1,0), (0,2,-1) , (1,1,2)} is linearly independent.

2. Is the set of polynomials $\{x+1, x^2-2, x^2-3x-5\}$ linearly dependent? Justify your answer.

3. Can you find a linear combination of (4,-2,1) and (-3,1,2) that equals the vector, (6,-4, 7)? If so, give the coefficients. If not, prove it.

Linear Transformations

A *linear transformation* is a function between vector spaces, $T:V \to W$, such that

$$T(a_1 v_1 + a_2 v_2) = a_1 T(v_1) + a_2 T(v_2)$$

This means that linear transformations preserve the linear combinations of vectors. In particular, if the linear transformation is one-to-one, then the linear transformation sends a basis of *V* to a basis of *W*. You will study this more in depth in Linear Algebra, so we will only cover the more important theorems for Differential Equations, and leave most of the proofs to your Linear Algebra course.

All linear transformations can be written as matrices. We have been exposed to matrices with real coefficients in the past, and they have usually been linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. In general, if we have a linear transformation, $T:V \to W$, between two vector spaces, V and *W* with respective bases $\{v_1, v_2, \cdots v_n\}$ and $\{w_1, w_2, \cdots, w_m\}$, we can create a matrix

$$\begin{array}{c} \\ w_1 \\ w_2 \\ \vdots \\ \vdots \\ w_m \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & \cdots \; \cdots & v_n \end{array} \\ \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ & & & \\ a_{m1} & a_{m2} & & a_{m3} \end{bmatrix} \end{array}$$

where

Theorem: A linear transformation, $T:V \to W$,is one-to-one if and only if the following are true:

a) $T(v) = 0$ implies $v = 0$.

b) det(T) $\neq 0$

c) The column vectors and row vectors are linearly independent.

d) The matrix row reduces to the identity matrix.

The proof of this theorem will be done in Linear Algebra.

Problem:

1) Show that the composition of linear transformations is a linear transformation.

2) Let $T:\mathbb{R}[x]\to\mathbb{R}[x]$ be defined by $T(p(x)) = p'(x)$. Show that it's a linear transformation.

3) Let $L:\mathbb{R}[x]\to\mathbb{R}[x]$ be defined by $L(p(x)) = xp(x)$. Let T be defined as above. Compute $L\circ T(p(x))$ and $T\circ L(p(x))$.