Topics: Logistic Differential Equations

Goals: Students will be able to model problems with differential equations and use current data to write a logistic model for the data.

Daily Recommended Schedule:

Monday: Read summary and do Problem 1
Tuesday: Do Problems 2 and 3 and read Problem 4.
Wednesday: Do Problem 4a
Thursday: Do Problem 4b.
Friday: Do Differential Equation Review.

Optional:

If you have access to the internet and would like to help me test this out, please, click on the link below corresponding to each day and try the check for understandings.

Monday
Tuesday
Wednesday

Resources:

Differential Equations Cheat Sheet (end of packet)
Review of Partial Fractions (www.baileyworldofmath.org)
Review Lesson on Euler's Method (www.baileyworldofmath.org)
Khan Academy Video on Separable Differential Equations
Khan Academy Videoson Logistic Differential Equations

MONDAY: Although a population grows exponentially at first, it cannot outgrow its resources without consequences. For this reason, Verhust, a Belgian mathematician modeled population growth with a more accurate differential equation given by the theory that the rate of population growth is proportional to the population and its difference to the limiting population.

This is modeled by the differential equation:

$$\frac{dP}{dt} = kP(L-P)$$

Where the population function is given by P(t) at time t with limiting population, L.

Example 1:

A Petri dish has a carrying capacity of 24 micrograms of bacteria. If a scientist initially introduced 6 micrograms of bacteria and the proportionality factor is 3 micrograms per hour.

Write the differential equation: (Use the limiting population (carrying capacity) and proportionality factor:

$$\frac{dP}{dt} = 3P(24-P)$$

Separate the Variables and Integrate both Sides (You will need partial fractions):

$$\frac{dP}{dt} = 3P(24-P)$$

$$\frac{dP}{P(24-P)} = 3dt$$

$$\frac{\frac{1}{24}}{P} + \frac{\frac{1}{24}}{24-P} = 3dt$$

$$\frac{1}{24}\left(\frac{1}{P} + \frac{1}{24-P}\right) = 3dt$$

$$\frac{1}{24}\int \frac{1}{P} + \frac{1}{24-P} = \int 3dt$$

$$\frac{1}{24}\left(\ln|P| - \ln|24-P|\right) = 3t + C$$

$$\frac{1}{P(24-P)} = \frac{A}{P} + \frac{B}{24-P}$$

$$1 = A(24-P) + BP$$

$$1 = 24A \qquad \text{if } P = 0$$

$$\frac{1}{24} = A$$

$$1 = 24B \qquad \text{if } P = 24$$

$$\frac{1}{24} = B$$

Use the initial condition: At $t = 0$, $P = 6$

$$\frac{1}{24}\left(\ln|6| - \ln|24-6|\right) = 3(0) + C$$

$$\frac{1}{24}(\ln(6) - \ln(18)) = C \quad \Rightarrow \quad \frac{1}{24}\ln(\frac{6}{18}) = C \quad \Rightarrow \quad \frac{1}{24}\ln(\frac{1}{3}) = C$$

Solve for P(t): (continued)

$$\frac{1}{24}(\ln(P)-\ln(24-P))=3t+\frac{1}{24}\ln(\frac{1}{3})$$

$$\ln(\frac{P}{24-P})=24(3t+\frac{1}{24}\ln(\frac{1}{3}))$$

$$\ln(\frac{P}{24-P})=72t+\ln(\frac{1}{3})$$

$$\frac{P}{24-P}=e^{72t+\ln(\frac{1}{3})}$$

$$\frac{P}{24-P}=e^{72t}e^{\ln(\frac{1}{3})}$$

$$\frac{P}{24-P}=\frac{1}{3}e^{72t}$$

$$P=(24-P)(\frac{1}{3}e^{72t})$$

$$P=8e^{72t}-\frac{P}{3}e^{72t}$$

$$P+\frac{P}{3}e^{72t}=8e^{72t}$$

$$P(1+\frac{1}{3}e^{72t})=8e^{72t}$$

$$P(t)=\frac{8e^{72t}}{1+\frac{1}{3}e^{72t}}$$

$$P(t)=\frac{8}{e^{-72t}+\frac{1}{3}}$$

$$P(t)=\frac{24}{3e^{-72t}+1}$$

<u>TUESDAY:</u> Make sure to show your work

A population of wolves is modeled by the function P and grows according to the logistic differential equation $\frac{dP}{dt} = 5P\left(1 - \frac{P}{5000}\right)$, where t is the time in years and $P(0) = 1000$. Which of the following statements are true?

I. $\lim_{t\to\infty} P(t) = 5000$

II. $\frac{dP}{dt}$ is positive for $t > 0$.

III. $\frac{d^2P}{dt^2}$ is positive for $t > 0$.

(A) I only
(B) II only
(C) I and II only
(D) I and III only
(E) I, II, and III

<u>Solution</u>: The differential equation is not of the form that immediately gives us the limiting population. To manipulate it to be the form, $\frac{dP}{dt} = \frac{5P}{5000}(5000 - P)$, so the limiting population is 5000, which means that $\lim_{t\to\infty} P(t) = 5000$. So I is true. Since logistic functions are always increasing, then II is true. We also know that the function has an inflection point, or is increasing fastest, when the population is half the limiting population, 2500. After the inflection point, $P(t)$ is concave down, and before which $P(t)$ is concave up, so III is false. Therefore, the answer is (C)

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2. A population of javelinas is modeled by the function P and grows according to the logistic differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{3000}\right)$, where t is the time in years and $P(0) = 1000$. Which of the following is equivalent to $\lim_{t\to\infty} P(t)$?

A) 1500

B) 3000

C) 4500

D) 6000

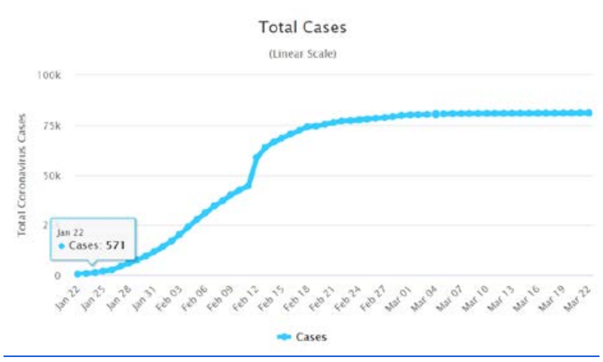
If we convert the differential equation to the form $\frac{dP}{dt} = kP(L - P)$ we should multiply both sides by 3000, $3000\frac{dP}{dt} = 3000P(2 - \frac{P}{3000}) = P(3000)(2 - \frac{P}{3000}) = P(6000 - P) \Rightarrow \frac{dP}{dt} = \frac{P}{3000}(6000 - P)$

Wednesday:

4. The Coronavirus Covid 19 is infecting people exponentially right now, but the model is a logistic equation because the limiting population is the population in the area in which the virus resides.

a)

CHINA



Selected Data

Date	Total Cases
1/22/2020	571
1/30/2020	9692
2/05/2020	28018
2/12/2020	58761
2/27/2020	78824
3/05/2020	80552
3/11/2020	80793
3/17/2020	80894
3/22/2020	81093

https://www.worldometers.info/coronavirus/country/china/

The graph above is a link to the actual data for the cumulative Covid19 cases in China. You can hover your mouse over the graph and find the actual numbers. Use some data points to find a logarithmic function to model the data above.

A solution: $P(t)$ will be a function in which P gives the total cases and t is the number of days after January 22. It seems that the limiting population is 82000. We don't really have an exact number, but this is a good estimate. This means that the differential equation satisfies the differential equation

$$\frac{dP}{dt} = kP(82000 - P)$$

Separate variables and integrate

$$\frac{dP}{dt} = kP(82000 - P)$$

$$\frac{dP}{P(82000 - P)} = kdt \quad \text{using partial fraction decomposition}$$

$$\frac{1}{82000}\left(\frac{1}{P} + \frac{1}{82000 - P}\right) = kdt$$

$$\int \frac{1}{P} + \frac{1}{82000 - P} = \int 82000kdt$$

$$\ln|P| + \ln|82000 - P| \quad = 82000kt + C$$

Since we have two unknowns, namely *k* and *C*, you need two data points. You can choose any two of the data points above, but I will choose $t = 0$, $P = 571$ and $t = 21$, $P = 58761$.

However, we can use a shortcut: the logistic equation cheat sheet says that a logistic function is of the form, $P(t) = \frac{L}{1-ke^{-at}}$. So when $t = 0$, $P = 571$ which means

$$P(0) = \frac{82000}{1-ke^{-a(0)}}$$

$$\frac{82000}{1-ke^{-a(0)}} = 571$$

$$\frac{82000}{1-k} = 571$$

$$\frac{82000}{571} = 1-k$$

$$-142.608 = k$$

When $t = 21$, $P = 58761$

$$P(21) = \frac{82000}{1+142.608e^{-a(21)}}$$

$$58761 = \frac{82000}{1+142.608e^{-a(21)}}$$

$$1+142.608e^{-21a} = \frac{82000}{58761}$$

$$142.608e^{-21a} = 1.3955-1$$

$$e^{-21a} = \frac{0.3955}{142.608}$$

$$e^{-21a} = 0.00277$$

$$-21a = \ln(0.00277)$$

$$a = \frac{-5.8889}{-21}$$

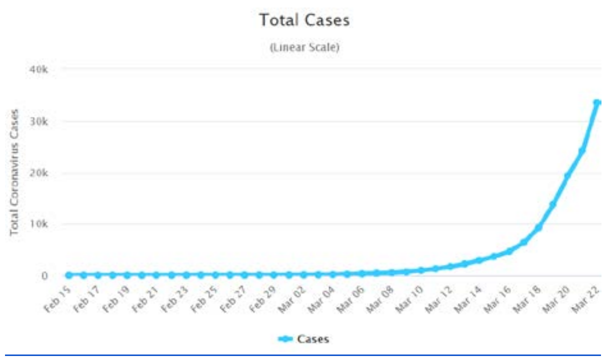
$$a = 0.280$$

So $P(t) = \frac{82000}{1+142.608e^{-0.280t}}$. If we want to make sure this is an accurate model, we plug in the current day and see how far away from the actual value our model is. For example, today is March 31, so it is 69 days since 1/22/2020 and $P(69) = 81999.952$. The actual value today, according to the website is 81518. So my model is somewhat accurate. Although using the y-intercept as a point is always useful, but you should try a different second point.

Thursday:

b)

UNITED STATES



Selected Data

Date	Total Cases
2/15/2020	15
2/21/2020	35
3/06/2020	319
3/12/2020	1697
3/16/2020	4663
3/18/2020	9259
3/20/2020	19383
3/22/2020	33566

https://www.worldometers.info/coronavirus/country/us/

The graph above is a link to Covid19 data for the US. Write a logarithmic model for this data. You may have to guess the limiting capacity. How close is your model to the actual data? Explain.

Coming soon: Use this and try different points to find k, a, and L. $P(t) = \dfrac{L}{1 - ke^{-at}}$

Friday: Differential Equation ReviewS

1) Consider the differential equation $\frac{dy}{dx} = y^2(2x+2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) Use Euler's Method, starting at $x = 0$ with two steps of equal size, to approximate $f(\frac{1}{2})$.

b) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

3) A certain rumor spreads through a community at the rate of $\frac{dy}{dt} = 2y(1-y)$, where *y* is the proportion of the population that has heard the rumor at time *t*. This is a logistic differential equation.

a) If at time *t* = 0 ten percent of the people have heard the rumor, find *y* as a function of *t*.

c) Find $\lim_{t \to \infty} y(t)$, and explain its meaning in the context of the problem.

Logistic Differential Equations

$$\frac{dy}{dx} = ay\left(1 - \frac{y}{L}\right) \text{ or } \frac{dy}{dx} = ay\left(\frac{L-y}{L}\right)$$

Partial Fraction Decomposition:	*Information about y*

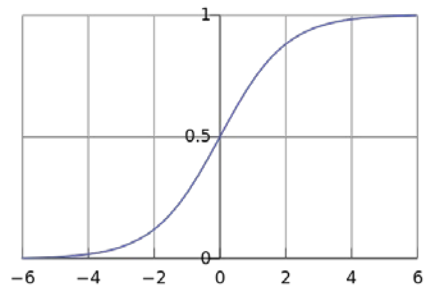
$$\frac{dy}{dx}=ay(1-\frac{y}{L})$$
$$\frac{dy}{dx}=ay(\frac{L-y}{L})$$
$$\frac{Ldy}{y(L-y)}=adx$$
$$\int \frac{1}{y}+\frac{1}{(L-y)}=\int adx$$
$$\ln(y)-\ln(L-y)=ax+C$$
$$\ln(\frac{y}{L-y})=ax+C$$
$$\frac{y}{L-y}=e^{ax+C}$$
$$\frac{L-y}{y}=Pe^{-ax}$$
$$1-\frac{L}{y}=-Pe^{-ax}$$
$$\frac{L}{y}=1+Pe^{-ax}$$
$$y=\frac{L}{1+Pe^{-ax}}$$

Models Population Growth

$$y=\frac{L}{1-Pe^{-ax}}$$

L is called the **limiting factor** or **limiting population** or **carrying capacity**

$$\lim_{x\to\infty} y(x)=L$$



$$C=\frac{L-y(0)}{y(0)}$$

$y(x)$ is growing x_1 fastest at where $y(x_1)=\frac{L}{2}$

<u>Differential Equations and Euler's Method</u>

- Use Implicit Differentiation to find $\frac{d^2y}{dx^2}$. Substitute what you have as $\frac{dy}{dx}$.
- Euler's Method uses a Δx and some steps.
  - o Use that $y_2=y_1+\frac{dy}{dx}|_{(x_1,y_1)}\,\Delta x$.
  - o Then use $y_3=y_2+\frac{dy}{dx}|_{(x_2,y_2)}\,\Delta x$
  - o Etc

(continued on next page)

- To solve Separable Differential Equations
  - o Separate the variables
    - ▪ y's and dy's on one side of the equal sign
    - ▪ x's and dx's on the other side of the equal sign
    - ▪ dy's and dx's are NOT allowed to be in the denominator when integrating
  - o Integrate both sides
  - o Don't forget the C
    - ▪ C should only be on the side with the x.
  - o Use Initial condition

        - This can be done before solving for y to find “C”
    - Solve for y
        - Sometimes y is called P and x is called t: don’t let this confuse you