2. Universal Objects and Versal Objects

One of the most important, and useful, characteristics of categories, is the idea of a universal object. Through these definitions, we can define the product, the fiber product, and the co-product of two objects in any category.

Definition 2. Let $\mathcal{C}$ be a category.
$U \in \mathrm{Obj}(\mathcal{C})$ is a *universally repelling object* in $\mathcal{C}$ if for any $A \in \mathrm{Obj}(\mathcal{C})$ there exists a **unique** morphism $f : U \to A$.
It is also called an *initial object*

A universally repelling object, therefore, is an object that maps uniquely to every other object in the category. The word unique here is very important. Below is an example of a universally repelling object that is rather easy to see.

Example 1. Let $\mathcal{C}$ = (Groups, group homomorphisms)
We need a group which can map to every other group in existence.
So we need to ask ourselves, what is the common characteristic of all groups?
The identity.
They all must have an identity to be a group.
The universally repelling object in this category, then, is the trivial group, which contains only the identity.
The unique morphism maps the identity to the identity.

Definition 3. Let $\mathcal{C}$ be category.
$V \in \mathrm{Obj}(\mathcal{C})$ is a *versally repelling object* in $\mathcal{C}$ if for any $A \in \mathrm{Obj}(\mathcal{C})$ there exists a morphism, not necessarily unique, $f : V \to A$.

The only difference here is the word "unique". There may be many maps from the versally repelling objects to every other object in the category. The following example will illustrate this.

Example 2. (Nonempty subsets of a set X, injective functions)
Let X be a set.
$\mathrm{Obj}(\mathcal{C}) = \mathcal{P}(X)$ and for any two $A, B \in \mathrm{Obj}(\mathcal{C})$, $\mathrm{Mor}(A, B) = \{f : A \to B \mid f(x) \neq f(y) \Leftrightarrow x \neq y\}$.
A singleton set $\{x\}$ is the versally repelling object because there will always be a injective map from it to any subset of X, although the map can map to any element of the set.

Definition 4. Let $\mathcal{C}$ be a category.
$U \in \mathrm{Obj}(\mathcal{C})$ is a *universally attracting object* in $\mathcal{C}$ if for any $A \in \mathrm{Obj}(\mathcal{C})$ there exists a **unique** morphism $f : A \to U$.
It is also called a *terminal object*.

A universally attracting object is an object to which every other object in the category maps uniquely. The word unique here is still very important. Below is an example of a universally attracting object that is rather easy to see.

Example 3.
(Nonempty subsets of a set X, embeddings functions) Let X be a set.
$\mathrm{Obj}(\mathcal{C}) = \mathcal{P}(X)$ and for any two $A, B \in \mathrm{Obj}(\mathcal{C})$, $\mathrm{Mor}(A, B) = \{f : A \to B \mid f(x) = f(y) \Leftrightarrow x = y\}$.

The whole set X is the universally attracting object because there will always be a injective map into it from any subset, although the map is not uniquely determined by the subset.

Definition 5. Let $\mathcal{C}$ be category.
$V \in \mathrm{Obj}(\mathcal{C})$ is a *versally attracting object* in $\mathcal{C}$ if for any $A \in \mathrm{Obj}(\mathcal{C})$ there exists a morphism, not necessarily unique, $f : V \to A$.

The only difference here is the word "unique". There may be many maps from the versally repelling objects to every other object in the category. The following example will illustrate this.

Example 4. Let $\mathcal{C}$ = (Topologies on a set X, continuous maps)
By this I mean that the objects consist of the same set with different topological structures.
We need a topology whose open sets will always have an open preimage.
So we need to ask ourselves, which sets are always open under any topology?
The trivial topology, the topology whose open sets consist of the whole set and the empty set.
If $f : X \to Y$ then $f^{-1}(Y) = X$ and $f^{-1}(\varnothing) = \varnothing$.
The versally attracting object in this category, then, is X with the trivial topology.
What is the versal repelling object?
Note that the map is not unique because composition with a permutation of the elements of X will still be continuous.

2.1. **Product and Co-Product.** Let $\mathcal{C}$ be a category. Its' *associated category* $\mathcal{C}'$ is the category whose objects are defined to be $f : A \to B$ morphisms f between objects in $A, B \in \mathrm{Obj}(\mathcal{C})$ morphisms are pairs $(g : A \to A', h : B \to B')$ between two objects $f : A \to B$ and $f' : A' \to B'$ such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{g} & A' \\ f\big\downarrow & & f'\big\downarrow \\ B & \xrightarrow{h} & B' \end{array}$$

2.1.1. *Product.* In order to define the *product* of two objects A, B in a category $\mathcal{C}$, we need to create a new category $\mathcal{D}$ whose objects are objects $Y \in \mathrm{Obj}(\mathcal{C})$ with morphisms $p_1 : Y \to A$ and $p_2 : Y \to B$, and morphisms are commutative diagrams:

The product of A and B is the universally attracting object in this category. In other words, it is the object $A \times B$ with maps $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ such that for any other object $Y \in \mathrm{Obj}(\mathcal{C})$ with morphisms $p_1 : Y \to A$ and