Alternating Series

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Alternating Series

A series in which terms are alternately positive and negative is called an **Alternating series**

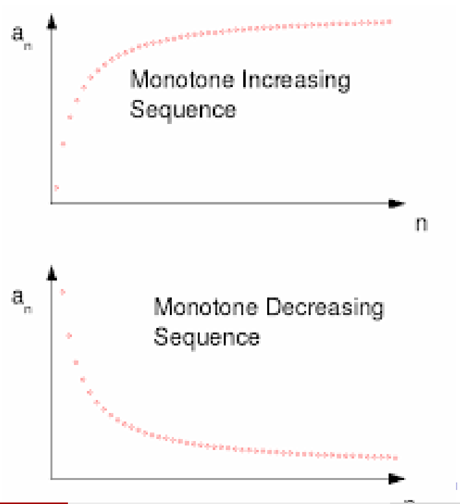
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots + \frac{1}{5} - \frac{1}{6} + \cdots$$

The **Alternating Harmonic Series** is much more well behaved than the regular Harmonic Series.

Before we study alternating series

Monotonic

Monotonic means stricly increasing or strictly decreasing.

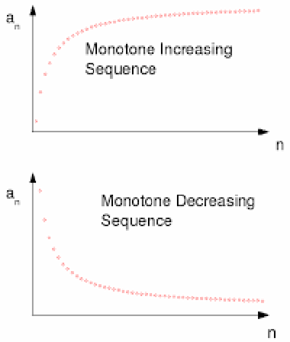


But wait, there's more!

A sequence which is strictly increasing, but has an upper bound, will have a limit.
A sequence which is strictly decreasing, but has a lower bound, will have a limit.

Bounded Monotonic

A bounded, monotonic sequence will always converge.



Back to Alternating Series

Sequence of Partial Sums in an Alternating Series

Some alternating series have a bounded, monotonic sequence of partial sums.

Let's look back at the Alternating Harmonic Series.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots + \frac{1}{5} - \frac{1}{6} + \cdots$$

The Sequence of Even Partial Sums is Bounded above

First, we will analyze the **even** partial sums.
The second partial sum is 1 minus something positive:

$$S_2 = 1 - \frac{1}{2} = \frac{1}{2} < 1$$

Since $\frac{1}{2} - \frac{1}{3}$ is also positive, then

$$S_4 = 1 - (\frac{1}{2} - \frac{1}{3}) - (\frac{1}{4}) = \frac{7}{12} < 1$$

the fourth partial sum is still 1 minus something positive.

$$S_6 = 1 - (\frac{1}{2} - \frac{1}{3}) - (\frac{1}{4} - \frac{1}{5}) - \frac{1}{6} < 1$$

The sequence of even partial sums is always less than 1.

Looking at the Even Partial Sums through a different Lens

Let's look at the even partial sums a different way: The second partial sum:

$$1 - \frac{1}{2} = S_2$$

The fourth partial sum:

$$\begin{aligned} S_4 &= (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) \\ &= S_2 + (\frac{1}{3} - \frac{1}{4}) \\ &> S_2 \end{aligned}$$

So the fourth partial sum is greater than the second partial sum.
In the next slide, we will see if the sixth partial sum is greater than S_4.

An increasing sequence

The sixth partial sum:

$$\begin{aligned} S_6 &= (1 - \frac{1}{2}) + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) \\ &= S_4 + (\frac{1}{5} - \frac{1}{6}) \\ &> S_4 \end{aligned}$$

Each subsequent even partial sum adds a positive number to the previous partial sum, thereby, increasing the sequence.
This means the sequence is strictly increasing and bounded above by 1, and, therefore, converges.

So the even partial sums converge, so what?

We now know that the sequence of even partial sums is an increasing sequence, bounded above by 1, so it must converge.

But what about the odd partial sums?

What do they do?

Why are we looking at every other partial sum?

I hope to answer these questions in the next few slides.

The odd partial sums are decreasing

Now let us see what happens to the odd partial sums:
The first partial sum

$$S_1 = 1$$

The third partial sum is

$$S_3 = 1 - (\frac{1}{2} - \frac{1}{3})$$

The fifth partial sum is

$$S_5 = 1 - (\frac{1}{2} - \frac{1}{3}) - (\frac{1}{4} - \frac{1}{5})$$

Since the sequence is decreasing because every odd partial sum takes away a positive number from the previous partial sum.

Another Lens?

Or we can look at it this way:
The first partial sum is positive:

$$S_1 = 1 > 0$$

The third partial sum is the sum of two positive numbers, and is therefore, positive:

$$S_3 = (1 - \frac{1}{2}) + \frac{1}{3} > 0$$

The Odd Partial Sums are Bounded below

The fifth partial sum is the sum of three positive numbers, and is therefore positive.

$$S_5 = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + \frac{1}{5} > 0$$

Each partial sum is still the sum of positive numbers, and is therefore positive.

Therefore, the odd partial sums are a monotonic decreasing sequence which is bounded below by zero, and therefore converges.

Okay. So the odd partial sums converse, so what?

Now we know that the even partial sums are increasing but bounded above by 1, so it converges.
And we know that the odd partial sums are decreasing and bounded below by 0, so it converges.
So both sequences converge, but do the converge to the same number?

But how do we know they have the same limit?

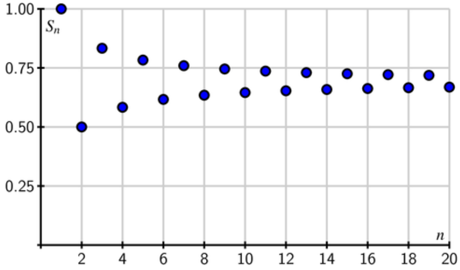
Let's look at what happens to the difference between an odd partial sum and the subsequent even partial sum:

$$\begin{aligned}
&\lim_{n\to\infty} |s_{2n+1} - s_{2n}| \\
=&\lim_{n\to\infty} |(1 - \frac{1}{2} + \cdots + \frac{1}{2n} - \frac{1}{2n+1}) - (1 - \frac{1}{2} + \cdots + \frac{1}{2n})| \\
=&\lim_{n\to\infty} \frac{1}{2n+1} \\
=&0
\end{aligned}$$

because the difference between consecutive partial sums is just the last term of the sequence.
Therefore, the odd and even partial sums converge to the same limit.

Sequence of Partial Sums of Alternating Harmonic Series

If we were to graph the partial sums, the graph would look like this.



Note that the only thing needed to make the alternating harmonic series converge was that:

- It alternated between positive and negative terms
- The absolute value of the terms were decreasing
- The sequence converged to zero

Alternating Series Test: Leibniz's Test

THEOREM 14—The Alternating Series Test (Leibniz's Test) The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. The positive u_n's are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

Sequence of Partial Sums of Alternating Series

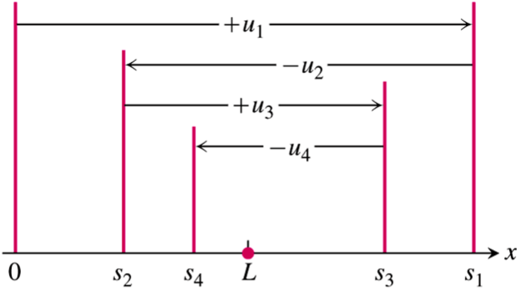


FIGURE 10.13 The partial sums of an alternating series that satisfies the hypotheses of Theorem 14 for $N = 1$ straddle the limit from the beginning.

Example 1

Does the series below converge?

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

Solution

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

By Alternating Series Leibniz Test, we need to check

- The sequence is alternating: This is given by the $(-1)^k$.
- The absolute value of the terms is decreasing:
  $\dfrac{1}{2k+1} > \dfrac{1}{2(k+1)+1}$ is a decreasing sequence
- The sequence converges to 0: $\lim_{k\to\infty} \dfrac{1}{2k+1} = 0$.

Therefore the series converges.

Check for Understanding 1

Do this series converge

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$$

- alternating: Yes, $(-1)^{k+1}$.
- absolute value decreasing: $\dfrac{1}{\sqrt{k}} > \dfrac{1}{\sqrt{k+1}}$
- converges to 0: $\lim_{k\to\infty} \dfrac{1}{\sqrt{k}} \to 0$

Check for Understanding 2

Do this series converge

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n)}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n)}$$

- alternating: Yes, $(-1)^n$.
- absolute value decreasing: $\ln(n)$ is increasing, so $\dfrac{1}{\ln(n)}$ is decreasing.
- converges to 0: Yes $\dfrac{1}{\ln(n)} \to 0$ as $n \to \infty$

I know, I know.

You're going to ask:

Do all alternating series converge?

No, but many of those that did not converge as positive series, converge as alternating series.

Absolute or Conditional Convergence

If $\sum a_k$ is an alternating series and $\sum |a_k|$ converges, then we say $\sum a_k$ **converges absolutely**.
If $\sum |a_k|$ diverges but $\sum a_k$ converges, then we say $\sum a_k$ **converges conditionally**.
It is clear that if $\sum |a_k|$ converges, then $\sum a_k$ converges.

Let's look back at our previous series.
Which converge conditionally, and which converge absolutely?

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n)}$
- $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Let's look back at our previous series.
Which converge conditionally, and which converge absolutely?

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges conditionally
- $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ converges conditionally
- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely

Alternating Series Rearrangement Theorem

THEOREM 17—The Rearrangement Theorem for Absolutely Convergent Series If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

How close is the n^{th} partial sum to the series

What is the difference between the n^{th} partial sum and the actual sum?

$$\sum_{k=0}^{\infty}(-1)^k a_k - \sum_{k=0}^{n}(-1)^k a_k$$

will only be the sum of the terms after n.
We call this the remainder.

$$\pm(a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + a_{n+5} - a_{n+6} \cdots)$$

if we group the terms, we can see that the remainder is less than a_{n+1}

$$\pm(a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) - \cdots)$$

Alternating Series Error Bound

This just says that if an alternating series converges, then the n^{th} partial sum is within $|a_{n+1}|$ of the actual sum.

THEOREM 15—The Alternating Series Estimation Theorem If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 14, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $L - s_n$, has the same sign as the first unused term.

Practice

Estimate the value of the following convergent series with an absolute error of less than .001

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^5}$$

Practice

Estimate the value of the following convergent series with an absolute error of less than .001

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^5}$$

If we want the error to be less than 0.001, then we to find $k+1$ such that $a_{k+1} < 0.001$.

$$\left|\frac{(-1)^{k+1}}{(k+1)^5}\right| < \frac{1}{1000}$$

by taking the reciprocal of both sides, we see that

$$(k+1)^5 > 1000$$

So

$$k + 1 > \sqrt[5]{1000}$$

$$k > 3.981 - 1$$

$k = 3$ would work, which means that if we estimate the series with

$$-1 + \frac{1}{32} - \frac{1}{243}$$

is within 0.001 of the value of the series.