COMPLEX FUNCTIONS: BRANCHES OF INVERSE

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1. The polynomial z^n

Let $w = z^n$ for some $n \in \mathbb{Z}$.
Then $w = (re^{i\theta})^n = r^n e^{ni\theta}$.
It is clear from this that $z = r^{\frac{1}{n}} e^{i\theta}$ is a solution.
However, so is $z = r^{\frac{1}{n}} e^{i(\theta + 2\frac{\pi}{n})}$.

Hence, z^n is a mapping from the complex z-plane to the complex w-plane, which wraps the punctured disc around itself 5 times and then is glued together at the ends. As we have already found, the n^{th} root function is a n-valued function for which each value is in one of the n sectors of equal area given by the following construction:
Let $S_0 = \{z \in \mathbb{C} \mid 0 \leq \arg(z) < \frac{2\pi}{n}\}$.
Let $S_1 = \{z \in \mathbb{C} \mid \frac{2\pi}{n} \leq \arg(z) < \frac{4\pi}{n}\}$
and $S_k = \{z \in \mathbb{C} \mid \frac{2k\pi}{n} \leq \arg(z) < \frac{2k\pi + \pi}{n}\}$.
Then $S_n = S_0$.
We can also view this phenomenon as n distinct functions,

$$\phi_k : \mathbb{C} \to S_k.$$

We call ϕ_0 the *principal branch of* n^{th} *root* .
Since winding around the origin in the z-plane results in 5 winds around the origin in w-space, it follows that the pull-back of one wind around the origin in w space would only be one-fifth of a wind in z-space. Hence, if we started at one branch of n^{th} root and wound around the origin in w-space, we would end up at the next branch of n^{th} root in z-space. Yet, we can avoid all of this confusion by omitting the portion of the domain space that causes all these problems. Namely, the positive real axis.
If we may a "cut " along the positive real axis, and prohibit ourselves from crossing this axis, then the branch problem is avoided. We call this a *branch cut.* We call the origin a *branch point* because winding around it causes us to move from one branch to another.

Can a similar analysis be made for $p(z) = (z - a)^n$, where a is a complex number?
Algebraically, if $w = re^{i\theta} = (z - a)^n$ then $z - a = |r|^{\frac{1}{n}} e^{i(\frac{\theta + 2\pi}{n})}$, which implies

$$z = |r|^{\frac{1}{n}} e^{i(\frac{\theta + 2\pi}{n})} + a.$$

Geometrically, we can compose p with a translation by a, $T_a(z) = z + a$ which maps bijectively and continuously onto the complex plane, and see that, topologically, the map acts identically on the plane with the exception that the branch point is no longer the origin, but a.

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2. The exponential function

The function $\exp(z) = e^z = e^{x+iy} = e^x e^{iy}$ is like z^n on steroids. Since $|\exp(z)| = e^{\Re(z)}$ and $arg(z) = \Im(z)$, we can see that every horizontal strip of length 2π maps to the entire punctured complex plane. In fact,for every k, the line, $L_k = \{a + 2\pi ik \mid a \in \mathbb{R}, k \in \mathbb{Z}\}$, maps to the positive real axis. So we can imagine that the exponential function rolls the plane into a cylinder which wraps around itself infinitely many times. Although, the fact that the length of the image vector increases exponentially would be lost by this interpretation. The exponential function is not surjective onto the complex plane because the length of the image is given by the exponential of the real part of z, which is never zero. Hence

$$\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$$

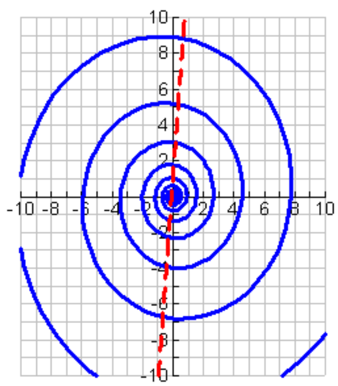
Recall, $\mathbb{C}$ is a group under addition, and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is a group under multiplication.
Since

$$\exp(z + w) = \exp(z)\exp(w)$$

then we can see that the exponential function is a group homomorphism, which means it sends addition to multiplication. The kernel of the map, or the preimage of the identity 1, is the set $\{2\pi ik \mid k \in \mathbb{Z}\}$. This implies that the exponential map is not an isomorphism, but

$$\exp : \mathbb{C}/ < 2\pi ik > \to \mathbb{C}^*$$

is an isomorphism.



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3. The Inverse of the exponential function

To find an inverse of $\exp(z)$, we set $\exp(z) = w$.
If $z = x + iy$ where $x, y \in \mathbb{R}$, then

$$w = e^x e^{iy}.$$

This yields

$$|w| = e^x \text{ which implies } x = \log(|w|)$$

and

$$\arg(w) = y.$$

However, $z' = x + iy + 2\pi i$ also maps to w via the exponential function. Hence for every w in the image space, there exist many $z \in \mathbb{C}$ in the domain space on a vertical line through $\log(|w|)$ equally spaced at intervals of 2π in the imaginary direction.

To show that $\log(z)$ has a branch point at 0, consider $z = re^{i\theta}$. Then

$$\begin{aligned}\log(z) &= \log(re^{i\theta}) \\ &= \log(r) + \log(e^{i\theta} \\ &= \log(r) + i\theta\end{aligned}$$

Now wrap around the origin once in the clockwise direction. This has increased you argument by 2π, but $z = re^{i(\theta+2\pi)}$, so

$$\log(z) = \log(r) + i(\theta + 2\pi).$$

Reiterating indefinitely will yield infinitely many solutions to $w = \log(z)$. Thus, wrapping around zero causes us to leap from one branch of log to another, without hope of ever returning.

We define the *principal branch of log* to be the branch for which the image of the positive real axis is real. In other words, the branch which maps to the set given by $D = \{x + iy | x \in \mathbb{R}\,,\, 0 \le y < 2\pi\}$.

4. Branch of Inverse

This leads us to a discussion of the function z^a where a is a complex number. There are two ways to define this function.
First, one may think of $z = re^{i\theta}$, allowing us to write $z^a = r^a e^{ia\theta}$.
Unfortunately, this does not help us understand the function much, so one may also allow $a = c+id$. Then we could write $z^a = r^{c+id}e^{i(c+id)\theta}$. By distribution, we achieve $z^a = r^c r^{id} e^{ic\theta - d\theta} = r^c(e^{-d\theta})r^{id}e^{ic}$. As you can see, this does little to help us understand the mapping. It is more common to define $z^a = \exp(a\log(z)) = \exp(a(\log(r) + i\theta)))$, where $\log(z)$ is the principal branch of logarithm. If $a = x + iy$, one can also view them as

$$\exp(a\log(z)) = \exp((x + iy)(\log(a))) = e^{x\log(a)}e^{i\log(a)}.$$

Do we need a branch of inverse for z^a?
If so, what would it be?

We will soon find that all multi-valued functions can be described in terms of branch of log. For instance,

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

If $w = cos(z)$ and $\log : \mathbb{C}^* \to D$ is the principal branch of log mapping to D as defined above, then

$$\begin{aligned} w &= \cos(z) \\ w &= \frac{e^{iz} + e^{-iz}}{2} \\ 2we^{iz} &= (e^{iz})^2 + 1 \\ 0 &= (e^{iz})^2 - 2we^{iz} + 1 \end{aligned}$$

This is a quadratic equation with roots

$$\begin{aligned} e^{iz} &= \frac{1}{2}2w \pm \sqrt{4w^2 - 4} \\ e^{iz} &= w \pm \sqrt{w^2 - 1} \\ iz &= \log(w \pm \sqrt{w^2 - 1}) \\ z &= -i\log(w \pm \sqrt{w^2 - 1}) \end{aligned}$$

Therefore,

$$\arccos(w) = -i\log(w \pm \sqrt{w^2 - 1}).$$

We can also see that

$$(w + \sqrt{w^2 - 1})(w - \sqrt{w^2 - 1}) = w^2 - (w^2 - 1) = 1$$

and, hence, are multiplicative inverses. Since we have the property

$$a = b^c \text{ then } \frac{1}{a} = b^{-c},$$

this implies

$$\arccos(w) = \pm i\log(w + \sqrt{w^2 - 1}).$$

This last result reflects the evenness of $\cos(z)$, and the branch of log reflects the periodicity of cosine.

5. Sources

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Problem 1. Express arcsin(w) in terms of log. [Hint: See if you can show that $sin(z) = cos(\frac{\pi}{2} - z)$.]

Problem 2. Express arctan(w) in terms of the principal branch of log.

Problem 3. Show that these functions are differentiable, and find their derivative.
You don't have to prove the Cauchy-Riemann Equations hold. You can go the more elegant route of using the derivative properties.

(a) $\log(z)$
(b) $\arccos(z)$
(c) $\arcsin(z)$
(d) $\arctan(z)$
(e) z^a

Problem 4. How can one define the "angle" between two complex numbers? Use this to show that the locus to the equation $z^n = a$ form the vertices of a regular polygon.
This will lead us to conformal mappings and explicitly defined stereographic projection.

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