Complex Numbers

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Imaginary Numbers are not Imaginary

Once the world of mathematics accepted the real numbers as an accurate model of time, distance and other continuous variables, most people thought that they had no need for a larger set of numbers. However, differential equations which surfaced through problems in physics, made it clear that another model was required to solve some fundamental problems. For example, the most accurate model for electromagnetism requires two real numbers, or an ordered pair of real numbers. One is the measure the intensity of the electric field and one to measure the intensity of the magnetic field. So why not use points on the real plane? This model requires the multiplication of these ordered pairs, and there is no obvious way to multiply points on the real plane.

The Definition of i

Definition

i is defined by the equation $i^2 = -1$

Therefore, $x^4 - 1 = 0$ which has two real roots $\{-1, 1\}$ now has four existant roots (the same numbers as the degree of the polynomial) $\{1, -1, i, -i\}$

The Complex Numbers

Definition (complex number)

A complex number is an element of the set $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$ We call a the real part of the complex number, and ib the imaginary part.

Right away we can see that these numbers have a geometric interpretation. If we consider the x-axis in the real plane, to be the set of real numbers, and the y-axis to be the imaginary numbers, we can graph every complex number on the plane such that $x + iy \dashrightarrow (x, y)$

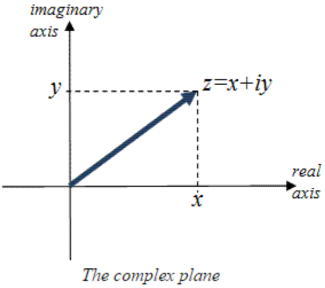


Figure: The x-axis becomes the real axis and y -axis becomes the imaginary axis

The Complex Module

Definition (module of z)

The *module* of $z = a + ib$ is the distance from z to the 0. This is analogous to the distance between (a, b) to the origin. So it makes sense that we would compute it the same way.

$$|z|^2 = a^2 + b^2$$

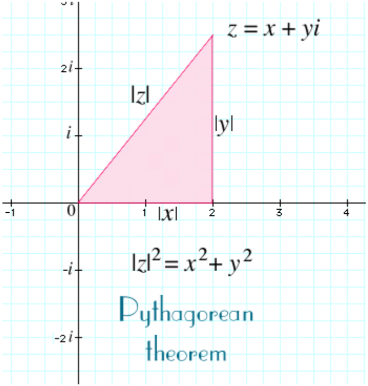


Figure: The length of z is a geometric idea which we will tie into complex multiplication later

On your whiteboards

Exercise

Graph and find the length of the following complex numbers:

- (1) $4 + 3i$
- (2) $4 - 3i$
- (3) $-2 + 5i$

The Complex Conjugate

Definition (complex conjugate)

The *complex conjugate* of $z = a + ib$ is the complex number $\bar{z} = a - ib$. Geometrically, it is the reflection of z over the real axis.

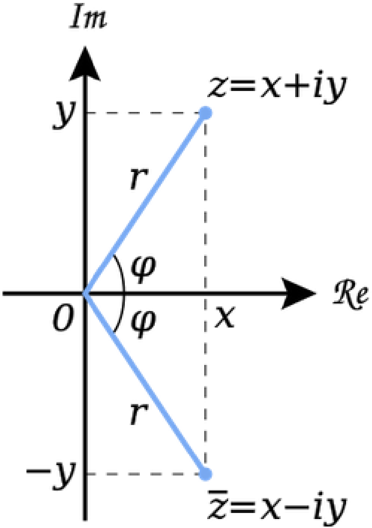


Figure: The conjugate of

Complex Addition

Let $z, w \in \mathbb{C}$ be complex numbers, such that $z = a + ib$ and $w = c + id$. Then we define the sum of z and w to be
$z + w = (a + ib) + (c + id) = (a + c) + i(b + d)$
the sum of the real parts, and the sum of the imaginary parts. (i acts like a variable here).

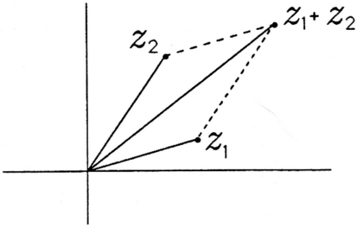
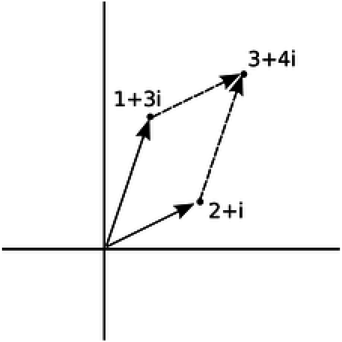


Figure: You can see that the sum of z and w is the diagonal of the parallelogram generated by z and w

Example

Below is an illustration showing the sum of $1+3i$ and $2+i$.

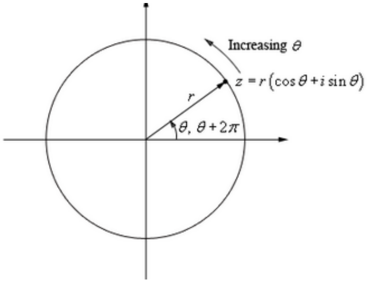


Exercise

On your whiteboards, draw and compute the sum of $5-2i$ and $3+2i$.

Sine and Cosine

Suppose we wanted to think of each point in the complex plane with respect to it's angle to the positive real axis and it's distance to the origin, rather than it's real and imaginary part.
We would want to write the real and imaginary part of a complex number as a function of the angle it makes with the positive real axis. Well, in order to figure out how to do it let us look at the simplest case.



MacLaurin Saves the Day

$$\begin{aligned} z &= a + ib \\ &= |z|\cos(\theta) + i|z|\sin(\theta) \\ &= |z|(\cos(\theta) + i\sin(\theta)) \end{aligned}$$

However, this notation is too long and since we have calculus on our side, we can simplify this even further.
Let $z = |z|(\cos(\theta) + i\sin(\theta))$.

Polar Form of Complex Number

$$\begin{aligned}
z &= |z|[\cos(\theta) + i\sin(\theta)] \\
&= |z|[\sum_{n=0}^{\infty}(-1)^n \frac{(\theta)^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}(-1)^n \frac{(\theta)^{2n+1}}{(2n+1)!}] \\
&= |z|[(1 - \frac{(\theta)^2}{2!} + \frac{(\theta)^4}{4!} - \cdots) + i(\theta - \frac{(\theta)^3}{3!} + \frac{(\theta)^5}{5!} + \cdots)] \\
&= |z|[1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} \cdots] \\
&= |z|[1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots] \\
&= |z|[\sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}] \\
&= |z|e^{i\theta}
\end{aligned}$$

Complex Multiplication

Let us see what happens whenever we multiply any two complex numbers.
Let $z = |z|e^{i\theta}$
Let $w = |w|e^{i\alpha}$
Then

$$\begin{aligned} zw &= |z|e^{i\theta}|w|e^{i\alpha} \\ &= |z||w|e^{i\theta+i\alpha} \\ &= |z||w|e^{i(\theta+\alpha)} \end{aligned}$$

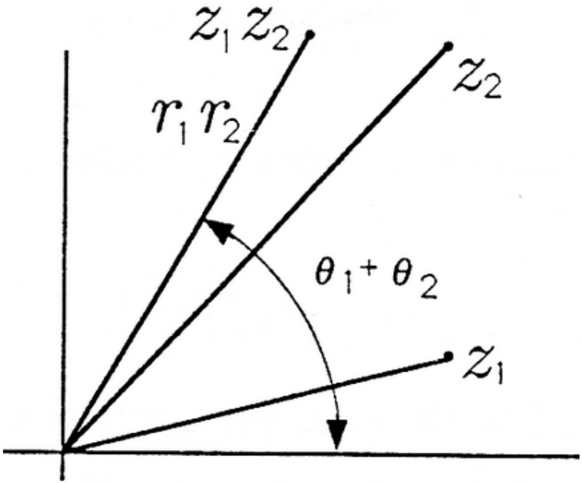


Figure: Complex multiplication is a rotation by the magnitude of a complex number along with a rotation by the angle given by the complex number.

Argument of a Complex Number

Definition (argument)

The argument of a complex number, $z = re^{i\theta}$, is the angle $\theta \in [0, 2\pi)$ the vector, z, makes with the positive real axis in the counter-clockwise rotation. We call r the module of z.
The argument of z is denoted $\text{Arg}(z)$.

Also, we can write the conjugate of a complex number in a different way now.
If $z = re^{i\theta}$, then $\bar{z} = re^{-i\theta}$.
From this perspective it is clear that $z\bar{z} \in \mathbb{R}$.
We can also find the multiplicative inverse of z rather easily.
Since $re^{i\theta}\frac{1}{r}e^{-i\theta} = e^0 = 1$, then we can see that $\frac{1}{z} = \frac{\bar{z}}{|z|}$.

Homework

Problem

Let $g : \mathbb{C} \to \mathbb{C}$ be defined by $g(z) = \bar{z}$.

- (a) Show $g(z + w) = g(z) + g(w)$. We say g preserves addition.
- (b) Show $g(zw) = g(z)g(w)$. We say g preserves multiplication.
- (c) Show that $g(\frac{1}{z}) = \frac{1}{g(z)}$. We say that g preserves multiplicative inverses.

The complex numbers are a *field*. A field is a set, F, with two binary operations, $(+, *)$, which we call *addition* and *multiplication* such that $(F, +)$ is a group and $(F\ \{0\}, *)$ is also a group. Since conjugation preserves addition and multiplication, we say it is a field *homomorphism*.

Problem

Let $C_{(r,z_o)} = \{z \in \mathbb{C} \big| |z - z_0| = r\}$ be the circle of radius r about z_0.

Let g be defined as in problem 1.

- (a) Show that $g(C_{(r,z_o)})$ is a circle.
- (b) Let $h(z) = kz$ where k is a fixed real number. Show $h(C_{(r,z_o)})$ is a circle.
- (c) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \dfrac{1}{z}$.
  Use parts (a) and (b) and that
  $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$ to show that $f(C_{(r,z_0)})$ is a circle

Problem

Let $z \in \mathbb{C}$, and $f(z) = \dfrac{1}{z}$.

- (a) Let $z = re^{i\theta}$.
  Let L be the line through the origin with angle θ with the positive x-axis.
  What happens to $f(z)$ as z moves through L.
- (b) Let C be a circle of radius r centered about the origin.
  What happens to $f(z)$ as z moves around C in the counter clockwise direction.
- (c) Let $D = \{z \in \mathbb{C} \mid |z| > 1\}$. Find the image $f(D)$.
- (d) Let $D' = \{z \in \mathbb{C} \mid |z| < 1\}$. Find the image $f(D')$.