MONODROMY ACTION:

I will try to help you to understand math by reading by scouring the Hatcher book with you.

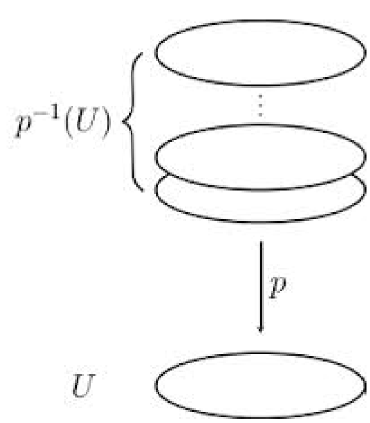
You will need your composition book.

Your homework will be to read and take notes.

Mar 1-11:08 PM

A Covering Space

Let us begin with the definition. A **covering space** of a space X is a space $\tilde{X}$ together with a map $p:\tilde{X}\to X$ satisfying the following condition: There exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p homeomorphically onto U_α. We do not require $p^{-1}(U_\alpha)$ to be nonempty, so p need not be surjective.



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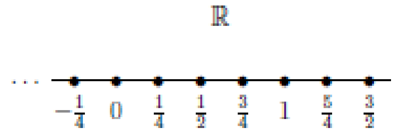
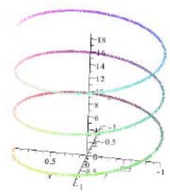
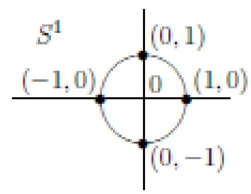
In the helix example one has $p:\mathbb{R}\to S^1$ given by $p(t) = (\cos 2\pi t, \sin 2\pi t)$, and the cover $\{U_\alpha\}$ can be taken to consist of any two open arcs whose union is S^1. A related example is the helicoid surface $S \subset \mathbb{R}^3$ consisting of points of the form $(s\cos 2\pi t, s\sin 2\pi t, t)$ for $(s,t) \in (0,\infty)\times\mathbb{R}$. This projects onto $\mathbb{R}^2 - \{0\}$ via the map $(x,y,z) \mapsto (x,y)$, and this projection defines a covering space $p:S\to\mathbb{R}^2 - \{0\}$ since for each open disk U in $\mathbb{R}^2 - \{0\}$, $p^{-1}(U)$ consists of countably many disjoint open disks in S, each mapped homeomorphically onto U by p.







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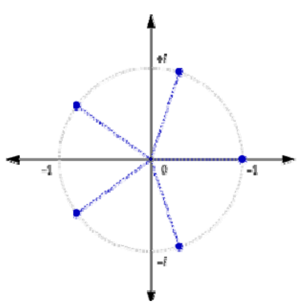
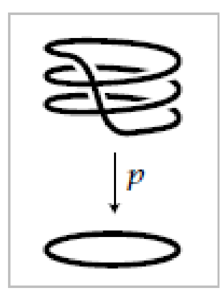
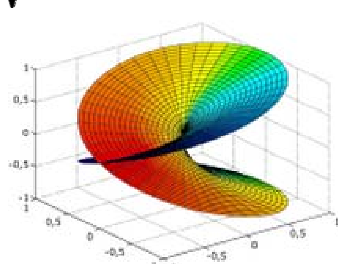
Another example is the map $p:S^1\to S^1$, $p(z) = z^n$ where we view z as a complex number with $|z| = 1$ and n is any positive integer. The closest one can come to realizing this covering space as a linear projection in 3-space analogous to the projection of the helix is to draw a circle wrapping around a cylinder n times and intersecting itself in $n - 1$ points that one has to imagine are not really intersections. For an alternative picture without this defect,







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$f(z) = z^3$

$f: \mathbb{C} \to \mathbb{C}$ is a function

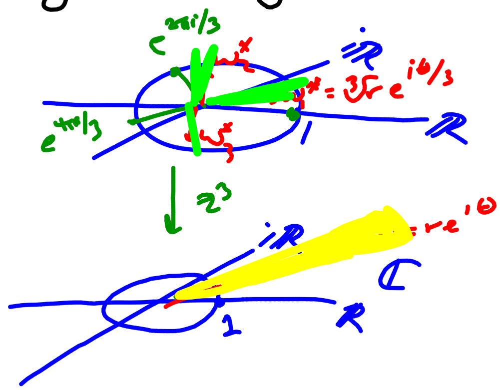
f is a 3:1 function everywhere except 0

If we want to see the geometry more clearly,

$z = re^{i\theta}$

$\downarrow$

$z^3 = r^3 e^{3i\theta}$



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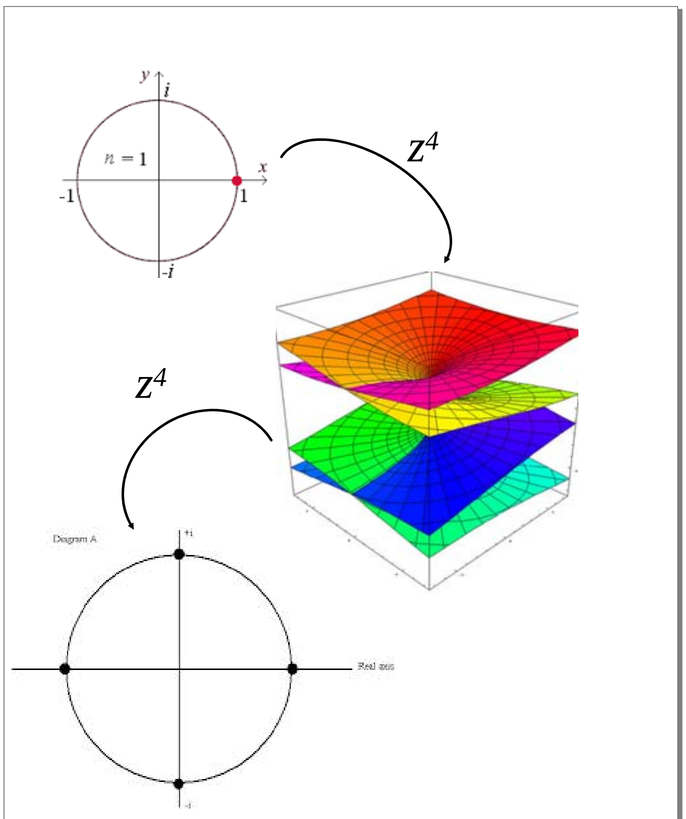

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Example: Define $p(z) = z^3$

Show this is a 3:1 cover of the complex plane, ramified over the origin.

Step 1: a. Draw the Base Space.

b. Find the Ramified Point

c. Find the Base Point.

d. What is the fundamental group.

Step 2: a. Draw the Covering Space

b. Find the fiber over the base point.

c. Find the path lifts at each point in the fiber.

Step 3: How does the Fundamental Group act on the fiber over the base point.

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Lifting Properties

Covering spaces are defined in fairly geometric terms, as maps $p:\tilde{X}\to X$ that are local homeomorphisms in a rather strong sense. But from the viewpoint of algebraic topology, the distinctive feature of covering spaces is their behavior with respect to lifting of maps. Recall the terminology from the proof of Theorem 1.7: A **lift** of a map $f:Y\to X$ is a map $\tilde{f}:Y\to\tilde{X}$ such that $p\tilde{f}=f$. We will describe three special lifting properties of covering spaces, and derive a few applications of these.

Proposition 1.30. *Given a covering space $p:\tilde{X}\to X$, a homotopy $f_t:Y\to X$, and a map $\tilde{f}_0:Y\to\tilde{X}$ lifting f_0, then there exists a unique homotopy $\tilde{f}_t:Y\to\tilde{X}$ of $\tilde{f}_0$ that lifts f_t.*

Proof: For the covering space $p:\mathbb{R}\to S^1$ this is property (c) in the proof of Theorem 1.7, and the proof there applies to any covering space. □

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Taking Y to be a point gives the **path lifting property** for a covering space $p:\widetilde{X}\to X$, which says that for each path $f:I\to X$ and each lift $\widetilde{x}_0$ of the starting point $f(0)=x_0$ there is a unique path $\widetilde{f}:I\to\widetilde{X}$ lifting f starting at $\widetilde{x}_0$. In particular, the uniqueness of lifts implies that every lift of a constant path is constant, but this could be deduced more simply from the fact that $p^{-1}(x_0)$ has the discrete topology, by the definition of a covering space.

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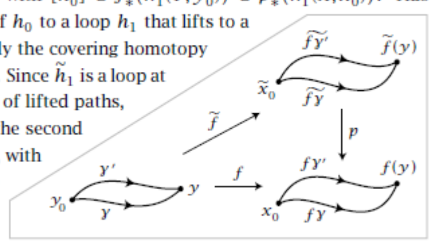
Proof: The 'only if' statement is obvious since $f_* = p_*\widetilde{f}_*$. For the converse, let $y\in Y$ and let γ be a path in Y from y_0 to y. The path $f\gamma$ in X starting at x_0 has a unique lift $\widetilde{f\gamma}$ starting at $\widetilde{x}_0$. Define $\widetilde{f}(y)=\widetilde{f\gamma}(1)$. To show this is well-defined, independent of the choice of γ, let γ' be another path from y_0 to y. Then $(f\gamma')\cdot(\overline{f\gamma})$ is a loop h_0 at x_0 with $[h_0]\in f_*(\pi_1(Y,y_0))\subset p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$. This means there is a homotopy h_t of h_0 to a loop h_1 that lifts to a loop $\widetilde{h}_1$ in $\widetilde{X}$ based at $\widetilde{x}_0$. Apply the covering homotopy property to h_t to get a lifting $\widetilde{h}_t$. Since $\widetilde{h}_1$ is a loop at $\widetilde{x}_0$, so is $\widetilde{h}_0$. By the uniqueness of lifted paths, the first half of $\widetilde{h}_0$ is $\widetilde{f\gamma'}$ and the second half is $\widetilde{f\gamma}$ traversed backwards, with the common midpoint $\widetilde{f\gamma}(1)=\widetilde{f\gamma'}(1)$. This shows that $\widetilde{f}$ is well-defined.



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Proposition 1.34. *Given a covering space $p: \tilde{X} \to X$ and a map $f: Y \to X$, if two lifts $\tilde{f}_1, \tilde{f}_2: Y \to \tilde{X}$ of f agree at one point of Y and Y is connected, then $\tilde{f}_1$ and $\tilde{f}_2$ agree on all of Y.*

Proof: For a point $y \in Y$, let U be an open neighborhood of $f(y)$ in X for which $p^{-1}(U)$ is a disjoint union of open sets $\tilde{U}_\alpha$ each mapped homeomorphically to U by p, and let $\tilde{U}_1$ and $\tilde{U}_2$ be the $\tilde{U}_\alpha$'s containing $\tilde{f}_1(y)$ and $\tilde{f}_2(y)$, respectively. By continuity of $\tilde{f}_1$ and $\tilde{f}_2$ there is a neighborhood N of y mapped into $\tilde{U}_1$ by $\tilde{f}_1$ and into $\tilde{U}_2$ by $\tilde{f}_2$. If $\tilde{f}_1(y) \neq \tilde{f}_2(y)$ then $\tilde{U}_1 \neq \tilde{U}_2$, hence $\tilde{U}_1$ and $\tilde{U}_2$ are disjoint and $\tilde{f}_1 \neq \tilde{f}_2$ throughout the neighborhood N. On the other hand, if $\tilde{f}_1(y) = \tilde{f}_2(y)$ then $\tilde{U}_1 = \tilde{U}_2$ so $\tilde{f}_1 = \tilde{f}_2$ on N since $p\tilde{f}_1 = p\tilde{f}_2$ and p is injective on $\tilde{U}_1 = \tilde{U}_2$. Thus the set of points where $\tilde{f}_1$ and $\tilde{f}_2$ agree is both open and closed in Y. ◻

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