LECTURE NOTES ON POLYNOMIALS I

L. MARIZZA A. BAILEY

We have known how to divide since the third grade (I hope).
We knew we could divide because of a special property owned by the set of integers.

Theorem 1 (The Division Algorithm)**.** *Given any two integers, $n, d \in \mathbb{Z}$, such that $d \leq n$. There exists $q, r \in \mathbb{Z}$ with $r < d$ such that*

$$n = dq + r$$

We call d the dividend*, q the* quotient*, and r the* remainder*.*

I will illustrate this concept with an example

Example 1. Let $n = 7$ and $d = 3$. These satisfy the conditions above. Then there exists $q = 2$ and $r = 1$ such that

$$n = qd + r7 \qquad = (3)(2) + 1$$

How on earth did I come up with those numbers, you ask?
Simple, I divided.

$$7 \div 3 = 2 \text{ Remainder } 1$$

Well, what does this have to do with polynomials? Fortunately, we the set of polynomials with real coefficients has the same property.

Theorem 2 (The Division Algorithm for the Polynomial Ring)**.** *Given any two polynomials, $p(x)$ and $q(x)$ such that $\deg(q) \leq \deg(p)$. There exists $q(x)$ and $r(x)$ with $\deg(r) < \deg(d)$ such that*

$$p(x) = d(x)q(x) + r(x)$$

We call $d(x)$ the dividend*, $q(x)$ the* quotient*, and $r(x)$ the* remainder*.*
NOTE: The remainder must have degree less than the dividend!!!!!

Let us illustrate with another example:

Example 2. Let $p(x) = x^3 - 3x^2 + 6x - 3$ and $d(x) = x^2 + 1$.

Date: October 09, 2006.

So, how does this help us?
Consider what would happen if we let $d(x) = x - a$, where a is any fixed number. and let $p(x)$ be any polynomial. Then we know there exists $q(x)$ and $r(x)$ such that

$$p(x) = (x - a)q(x) + r(x)$$

Remember ... $\deg(r) < \deg(x - a) = 1$. So $\deg(r) = 0$ or r is just a real number. Wait!!!
This means that if I plug a into p

$$p(a) = (a - a)q(a) + r = 0q(a) + r = r$$

In other words $p(a)$ is the remainder when I divide $p(x)$ by $x - a$. [Remainder Theorem]

That means, $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$. [Factor Theorem]

Department of Mathematics, Arkansas School of Mathematics, Sciences, and the Arts
E-mail address: baileym@asmsa.org