ROOTS OF UNITY

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1. Roots of Unity

1.1. **Complex Multiplication.** Previously, we learned that the product of $z = |z|e^(i\theta), w = |w|e^(i\alpha) \in \mathbb{C}$ is represented geometrically by the rotation of z by the angle $\alpha = \arg w$, which we will call the *argument of* z and dilation by $|w|$. Hence, we can write

$$zw = |z||w|e^{i(\theta+\alpha)}$$

1.2. **Powers of z.** Now we will look at the patterns of powers of a given complex number, z.

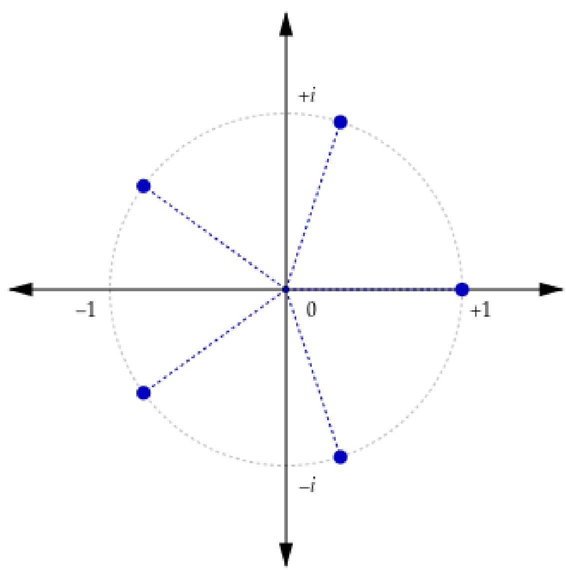
$$\begin{aligned} z^n &= (|z|e^{i\theta})^n \\ &= |z|^n e^{ni\theta} \end{aligned}$$

which means that powers of a complex number correspond to rotations by it's argument and dilations by it's length. Therefore, if $|z| = 1$, then

$$z^n = e^{ni\theta}$$

which implies that powers of a complex number on a unit circle is rotation by the argument of z.

Example 1. Let $z = e^{\frac{2i\pi}{5}}$.
The powers of z are illustrated below.



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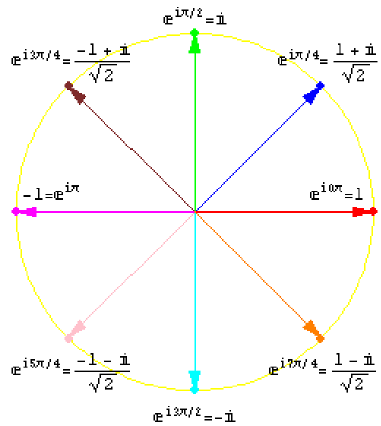
Example 2. Let $z = e^{\frac{2i\pi}{8}}$.
The powers of z are illustrated below.



1.3. **Roots of z.** How would I find a 4^{th} root of $5^4 i$?
Would my original complex number be returned?
Two equations would need to be satisfied.
The module equation:

$$r^4 = 5^4$$

and the argument equation:

$$4t = \frac{\pi}{2} + 2\pi k, \text{ where k is any integer}$$

The first equation is a little easier than the second.
Clearly, since $r \geq 0$, then $r = 5$ but there are many solutions to the second equation because angles are equivalent modulo 2π. The solution for

$$k = 0 \text{ is } \frac{\pi}{8}$$

$$k = 1 \text{ is } \frac{5\pi}{8}$$

$$k = 2 \text{ is } \frac{9\pi}{8}$$

$$k = 3 \text{ is } \frac{13\pi}{8}$$

All other integers would yield one of the above results. Hence, there are four fourth roots of $5^4 i$, or equivalently, there are four solutions to the equation $x^4 - 5^4 i = 0$. We know this last statement to be true by the Fundamental Theorem of Algebra.

1.4. **Roots of Unity.** We will, however, study a more interesting set of roots, *the roots of unity*, which are the solutions to the equation

$$x^n - 1 = 0$$

By the fundamental theorem of algebra, there should be n such solutions. We can find the *principal* root of unity rather easily,

$$x = e^{\frac{2\pi}{n}}$$

is a solution to the equation. This solution "generates" all the other solutions by taking powers of x.
Let $a = x^k$ for some $k \in \mathbb{Z}$.
Then

$$\begin{aligned} a^n - 1 &= (x^k)^n - 1 \\ &= (x^n)^k - 1 \\ &= (1)^k - 1 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

Now, we have found all the roots of unity, $\{e^{\frac{2k\pi}{n}} \mid k \in \mathbb{Z}\}$. Note that even though k runs through the integers, this set is finite.

2. Topology on the Complex Plane

2.1. **Distance.** Now that we have seen that one complex dimension corresponds to two real dimensions, it follows that the set given by

$$C(z_o, r) = \{z \,\Big|\, |z - z_o| = r\}$$

for a fixed $z_o \in \mathbb{C}$ and $r \in \mathbb{R}$ is a circle around z_o of radius r rather than an interval. To see this more clearly, recall that given $z = a + ib$, and $w = c + id$

$$\begin{aligned} |z - w| &= |a + ib - (c + id)| \\ &= |(a - c) + i(c - d)| \\ &= (a - c)^2 + (b - d)^2 \end{aligned}$$

This shows us that $|z-w|$ is nothing more than Euclidean distance defined on $\mathbb{R}^2$, which makes sense since $\mathbb{C}$ is essentially R^2 with a multiplication. Now we have a distance function on the complex plane and can, therefore, discuss when two complex numbers are close to each other. Since we will want to analyze complex functions, we will want to have an analogue to ϵ-intervals about a number for the complex plane.

2.2. **Neighborhoods.**

Definition 1 (image)**.** Let $f : X \to Y$ be a function from a set X to a set Y. The *image* of $A \subset X$ under a function f is defined to be is defined to be the set of all values of elements in the set A,
$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \in A\}$ and is denoted $f(A)$.

Definition 2 (pre-image)**.** Let $f : X \to Y$ be a function from a set X to a set Y.
The *pre-image* of $B \subset Y$ is the set of all values that are mapped to some element in B,
$\{f^{-1}(B) = \{x \in X \mid f(x) \in B\}$, and is denoted $f^{-1}(B)$.

Recall, in single real variable calculus, we often looked at the image of pre-image of intervals of the form $B_\epsilon = (x - \epsilon, x + \epsilon)$ under functions, f, to see whether they were continuous, differentiable or had limits. These intervals were always open, connected, and consisted of elements "close" to x.

Definition 3 (neighborhood)**.** A *neighborhood* of a complex number z, N_z, is an open disc of some positive radius, ϵ, around z,

$$N_z = \{w \in \mathbb{C} \Big| \, |z - w| < \epsilon \text{ for some } \epsilon \in (0, \infty)\}$$

Therefore, this neighborhood includes the point z but not the points on the outside of the circle (Churchill, 16)

A *deleted neighborhood* of a point z is a neighborhood with z removed, $N_z \backslash \{z\}$.

We will now use this definition to discuss the idea of limit.

2.3. **Limit Points.**

Definition 4 (limit point)**.** A point $a \in \mathbb{C}$ is a *limit point* of a set $X \subset \mathbb{C}$ if every deleted neighborhood of a, $D_a = N_a \setminus \{a\}$, has non-empty intersection with X,

$$X \cap D_a \neq \varnothing$$

This means that limit points of a set X can be inside of X or outside. We will do some examples.

Example 3. Let $X = \{z \Big| \, |z| < 1\} \cup \{2i\}$. What would be the set of limit points of X?
$2i$ definitely does not satisfy the condition because the deleted open disc of radius $\frac{1}{2}$ does not intersect with X. All the points inside the open unit disc are limit points because regardless of what radius I choose, their deleted neighborhoods will have nontrivial intersection with X.
Also, those points on the unit circle will also be limit points to X, for all the discs centered about points on the unit circle will intersect the open unit disc. Hence, the set of limit points of X consists of the closed unit disc $\{z \Big| \, |z| \leq 1\}$.

We call a set *closed* if it contains all of its limit points. Notice, it does not have to equal its set of limit points, only contain them. For those who took topology, I would like to emphasize the difference between limit points and boundary points (or adherent points). The boundary point definition did not require deletion.

2.4. **Interior.**

Definition 5 (interior point)**.** A point $x \in X$ is an interior point if there exists a neighborhood of x, N_x , such that $N_x \subset X$.

Indeed, if X is the closed unit disc, then all the points in the open unit disc are interior points. For those of you who took topology, I know this sounds like a circular definition, but recall, we are defining the open unit disc to be the set

$$\{z \in \mathbb{C} \Big| \, |z| < 1\}$$

However, in general, we call a set *open* all of it's elements are interior points.

2.5. **Path-connected.** The last property we need to discuss is *path connected.* A path in a set X is a function from the unit interval $I = [0, 1]$ to X, denoted $p : I \to X$.

Definition 6. We say a set X is path connected if for all points $x, y \in X$ there exists a path from x to y completed contained in X. Equivalently, there exists $p : I \to X$ such that

- **(a)** $p(0) = x$ and $p(1) = y$.
- **(b)** $p(I) \subset X$

2.6. **Domains, Regions and Sectors.** In our study of analytic functions we will be interested mostly in those subsets $D \subset \mathbb{C}$ which are open and connected, *open regions or domains.*

Of particular interest will be *sectors* which are like infinite pieces of pie. The sector from a point $z = re^{\alpha}$ to a point $w = se^{\beta}$ is the set
(assume $\arg(z) < \arg(w)$)

$$S = \{te^{\theta} \mid t \in [0, \infty) \text{ and } \theta \in (\alpha, \beta)\}$$

is the set of all points whose argument is between that of z and w.

Note that it is the allowance of t to be any non-negative real number which gives us the "ray" with argument θ. In fact, lines in the complex plane are written parametrically. The graph of the set, $L = \{z + wt \mid z, w \in \mathbb{C}\}$ is the line through the complex points z in the direction of w.

3. Problems

Problem 1. Practice graphing the following complex numbers, ~~their product~~ their product and their sum. Make sure you write them in polar form, as well. If they are already in polar form, write them in rectangular form.

- **(a)** $z = 2 + 2i$, $w = \sqrt{3} - i$.
- **(b)** $z = e^{\frac{\pi}{4}\mathrm{i}}$, $w = 3e^{\frac{3\pi}{8}\mathrm{i}}$.

Problem 2. Let $z \in \mathbb{C}$.
Show that $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$.

Problem 3.
- **(a)** Find, explicitly, the roots of the polynomial $x^{12} - 1$ in the complex plane.
- **(b)** Let $G = \{x \in \mathbb{C} \mid x^{12} - 1 = 0\}$.
  Show that
  - **(1)** $1 \in G$
  - **(2)** If $x \in G$ then $\dfrac{1}{x} \in G$.
  - **(3)** If $x, y \in G$ then $xy \in G$.

  These properties show that the twelfth roots of unity form a subgroup of $\mathbb{C}$ under multiplication.

Problem 4. Let $\mathcal{F} = \{f_w : \mathbb{C} \to \mathbb{C} \mid f_w(z) = wz\}$ be the family of functions defined by left multiplication.
Let $S = \{ke^{\theta} \mid k \in [0, \infty) \text{ and } \theta \in (0, \frac{\pi}{4})\}$ be the sector between the real-axis and
the point $1 + i$.
Let $w \in \mathbb{C}$ be a fixed point.
Find the image of S under the function f_w.
What happens to the image of S as w varies over $\mathbb{C}$?

Problem 5. **(a)** Let x be a solution to the equation $x^n - 1 = 0$.
Show $1 + x + x^2 + x^3 + \cdots + x^{n-1} = 0$.
(hint: Show $(1 - x)(1 + x + x^2 + x^3 + \cdots + x^{n-1}) = 1 - x^n$)

(b) Show if $U = \{a_1, a_2, a_3\}$ is the set of third roots of unity then

$$\sum_{i=1}^{3} a_i^2 = \sum_{i \neq j} a_i a_j$$

Note: This is the set of vertices of an equilateral triangle inscribed in the unit circle

(c) Let $u, v \in \mathbb{C}$ be fixed complex numbers. Then $U' = \{b_i = ua_i + v \mid a_i \in U\}$ is a rotation, dilation, and translation of the original set U, and therefore is also the set of vertices of an equilateral triangle. Show the equation in part (b) still holds true if you replace the a_i's with b_i's.

(d) The converse of this last statement is also true. Namely, that if

$$\sum_{i=1}^{3} b_i^2 = \sum_{i \neq j} b_i b_j$$

then $\{b_1, b_2, b_3\}$ form the vertices of an equilateral triangle.
To do this, I suggest you move this set with a transformation that will make your computation much easier. We know we can perform this transformation without losing generality by part (c). The beauty of this method is that you do not need to find the transformation explicitly, just know that one exists. You only have to find the right points that will reduce your problem to solving a quadratic equation.

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