UNIFORM CONTINUITY AND UNIFORM CONVERGENCE

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Abstract. We are familiar with the notions of continuity and convergence. A function is continuous if small movements in the domain produce small movements in the image. However, the range at which the image moves depends on the point in the domain. In other words, the *delta* depends on both x and *epsilon.* The properties of uniformity will deal with the situation in which the dependence on x is lifted.

1. Uniform Continuity

A function, f, is continuous if for all $a \in D$, where D is the domain of f,Given any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

. This x moved one-dimensionally over the real numbers, and over the complex number, we understand that $|x - a| < \delta$ is meant to be complex distance. An equivalent definition for continuity uses the concept of preimage.

Definition 1. *[Continuous] A function, $f : D \to \mathbb{C}$ is continuous if given any open ball $B_\epsilon(x) \subset \mathbb{C}$, it's preimage,*

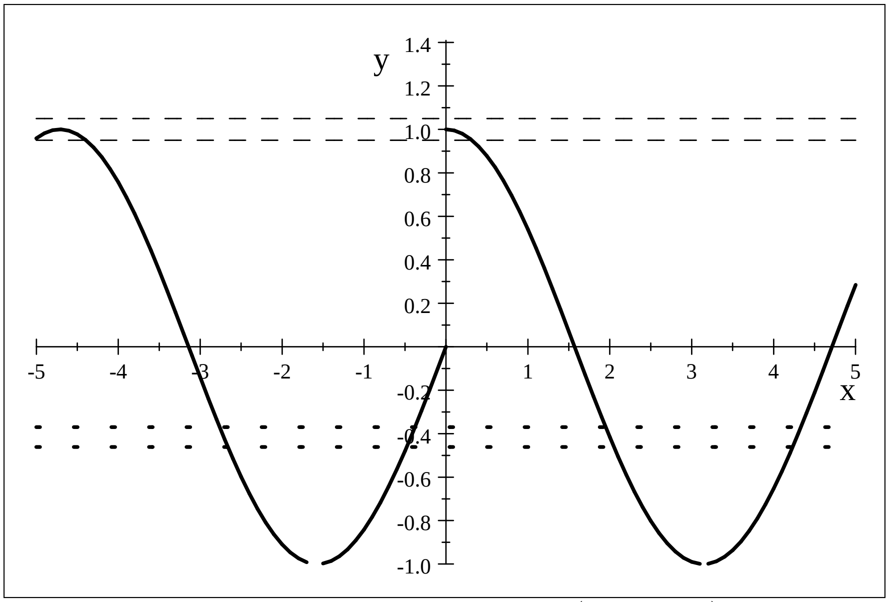
$$f^{-1}(B_\epsilon(x)) = \{z \in D \mid f(z) \in B_\epsilon(x)\}$$

is the union of open balls.

Note that over the real numbers, open balls are just open intervals. Below is a graph of a function continuous at a and discontinuous at b, to illustrate the equivalence of both definitions.

Example 1. *Let* $\begin{cases} f(x) & = \sin(x) \quad x \in (-\infty, 0) \\ & = \cos(x) \quad x \in [0, \infty) \end{cases}$ *.Then f(x) is continuous at x =2 and discontinuous at x = 0.*

$$f(x) = \begin{cases} \sin(x) & if \quad x < 0 \\ \cos(x) & if \quad x \geq 0 \end{cases}$$



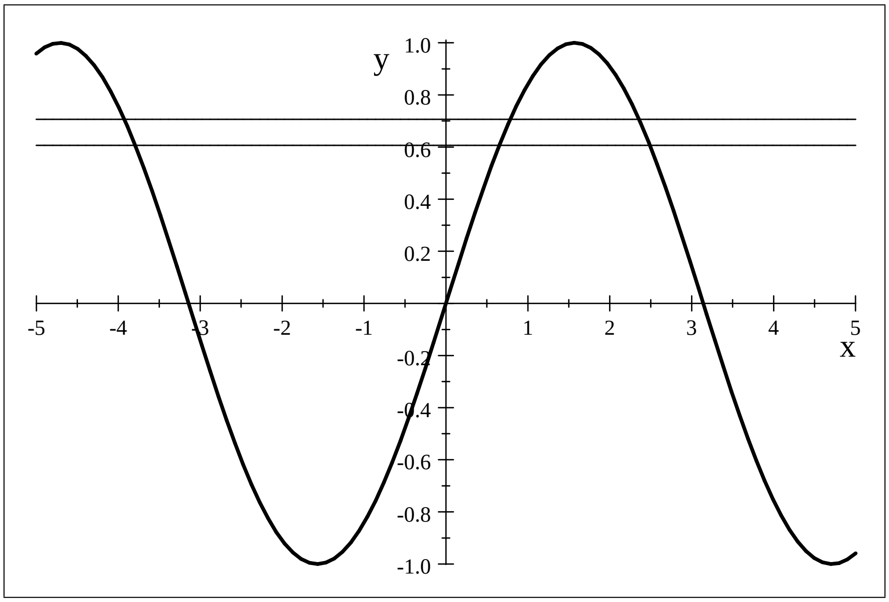
The preimage of the open interval $(y+\epsilon, y-\epsilon)$

Note, the preimage of the open interval $(1-\epsilon, 1+\epsilon)$,represented by the dashed lines, is the non-open interval, $[0, \arccos(1+\epsilon))$. Yet at every other point, in particular x = 2, then preimage of the open interval $(2-\epsilon, 2+\epsilon)$ is the union of open intervals. Mark the preimage of an open interval around the point y = 1 with red, and the preimage of the open interval about the point y = 2 with blue.

Definition 2. *[Uniformly Continuous] A function $f: D \to \mathbb{C}$ is uniformly continuous, if for any $\epsilon > 0$ there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < \epsilon$..*

Algebraically, this means that the delta does not depend on the x, but only on the size of epsilon.Geometrically, the pre-image of any ball of radius epsilon, $B_\epsilon(y)$, for any y in the range, will be the union of balls of radius delta about x, $B_\delta(x)$, such that $f(x) = y$.

Example 2. $f(x) = \sin(x)$



Since we cannot graph the complex counter part, we will analyze the real valued function, sin(x).The length of the pre-image of the epsilon neighborhood can be seen by the horizontal length of the graph which lies within the epsilon interval. Mathematically, pre-image of the intersection of the graph of f and the epsilon neighborhood is the union of the x-intervals which map to a connected component of the interesection. As you can see, these connected components have bounded horizontal length because the derivative is bounded. This means we can find a minimum delta that will work for all x. Geometrically, this is what it means to be uniformally continuous.

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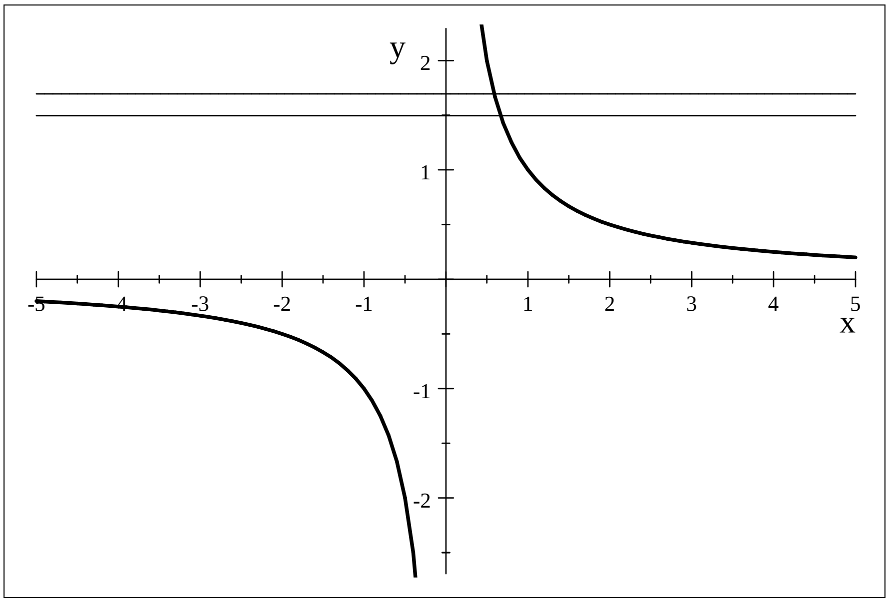
Example 3. $g(x) = \dfrac{1}{x}$.



As the epsilon neighborhood moves upwards, the horizontal length of the graph which li^{n}es within the epsilon neighborhood decreases to 0.The preimage is an open interval which implies the function is still continuous, however, there is no delta neighborhood of minimum length as we move across the y-axis, and thus one delta will not work for all x.

Let us see how to show that $\sin(x)$ is uniformly continuous analytically.

Proof. Let $\epsilon > 0$.

Then since the derivative of $\dfrac{d\sin(x)}{dx} = \cos(x)$, then there exists $\delta > 0$, such that $|x - y| < \delta$ implies

$$|\frac{\sin(x) - \sin(y)}{x - y} - cos(y)| < 1$$

Therefore

$$|\frac{\sin(x) - \sin(y)}{x - y}| - |\cos(x)| < 1$$

By adding one to both sides we achieve

$$\left|\frac{\sin(x) - \sin(y)}{x - y}\right| - |\cos(y)| + 1 < 2$$

Grouping together the right part of the left hand side,

$$\left|\frac{\sin(x) - \sin(y)}{x - y}\right| + (1 - |\cos(y)|) < 2$$

Since $|\cos(x)| \leq 1$, $1 - |\cos(y)| \geq 0$ and we get

$$|\frac{\sin(x) - \sin(y)}{x - y}| < 2$$

Multiplying both sides by $|x - y|$, we achieve,

$$|x - y||\frac{\sin(x) - \sin(y)}{x - y}| < |x - y|2$$

Hence, if $\delta < \frac{\epsilon}{2}$, the implication holds. □

Note that the boundedness of the derivative of $\sin(x)$ is crucial to the proof the $\sin(x)$ is uniformly continuous. This leads us to the following theorem.

Theorem 1. *If $f : D \longrightarrow \mathbb{C}$ is differentiable with bounded derivative, then f is uniformly continuous.*

Proof. Exercise. □

Exercise 1. *Show that $f(x) = \frac{1}{x}$ is not uniformly continuous.*

2. Convergence of functions

Suppose $(f_n(x))$ is a sequence of functions from a domain, D, to the complex plane.If we fix a point, $x \in D$, we get a sequence of complex numbers $(f_n(x))$. If the sequence converges at each x, then this gives us a new complex valued function on D . Here is an example of a sequence of functions. Can you identify which function it converges to?

Definition 3. *[pointwise convergence]A sequence of functions $f_n : X \rightarrow X$ converges to a function f if and only if for all $\epsilon > 0$ and each $x \in X$ there exists there exists N such that $n \geq N$ implies*

$$|f_n(x) - f(x)| < \epsilon$$

Example 4. *Let*

$$D = \{z \in \mathbb{C} \mid |z| \geq 1, \}$$

be the complement of the open unit disc.

Let $f_n(z) = \frac{1}{z^n}$. . Then for each $z \in D$, $f_n(z) = \frac{1}{z^n}$. This sequence converges to 0 if $|z| > 1$ and if $z \geq 1$, the sequence converges to a point on the unit circle.The function to which it converges is not continuous.

Example 5. *Let*

$$f_n : \mathbb{R} \rightarrow \mathbb{R}$$

be defined by $f_n(x) = \frac{1}{n}\sin(nx)$.
What function does this converge to pointwise?
Is it continuous?

Example 6. *Let*

$$f_n : \mathbb{R} \to \mathbb{R}$$

be defined by $f_n(x) = \sin^n(x)$.
What function does this converge to pointwise?
Is it continuous?

Definition 4. *[uniform convergence] A sequence of functions* $f_n : X \longrightarrow X$ *converges uniformly to* f *if and only if, for each*

$\epsilon > 0$ there exists $N > 0$ such that for all $x \in X$,

$$|f_n(x) - f(x)| < \epsilon$$

for all $n \geq N$. How does this differ from the definition of pointwise convergence? Which one of the above examples converges uniformly?

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