TRIG TECHNIQUES

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1. Trig Identities

Here are some examples that you may encounter while solving integrals with trigonometric functions.

Problem 1.

$$\int \sin^3(x)\cos^2(x)dx$$

Solution 1. If $\sin^3(x)$ were a $\sin(x)$, we could use u-substitution. Alas, it is not. So, we will substitute $1-\cos^2(x)$ for a $\sin^2(x)$ and leave the $\sin(x)$ as an odd man out. We will be left with a polynomial in $\cos(x)$, and a $\sin(x)dx$ term perfectly positioned to be a du for $u=\cos(x)$.

$$\begin{aligned}
\int \sin^3(x)\cos^2(x)dx &= \int \sin(x)(1-\cos^2(x))\cos^2(x)dx \\
&= \int \sin(x)(\cos^2(x)-\cos^4(x))dx \\
&= -\int u^2 - u^4 du \qquad \text{letting } u=\cos(x) \text{ and } du=-\sin(x)dx. \\
&= -\frac{u^3}{3}+\frac{u^5}{5}+C \\
&= -\frac{\cos^3(x)}{3}+\frac{\cos^5(x)}{5}+C
\end{aligned}$$

Problem 2.

$$\int \sin^2(x)\cos^2(x)dx$$

Solution 2. Unfortunately, writing $\sin^2(x)=1-\cos^2(x)$ will only leave us with a mess of $\cos^2(x)$ to which we would have to individually decrease the exponent by using the half angle trig identity.

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)dx &= \int \frac{1-\cos(2x)}{2}\cdot\frac{1+\cos(2x)}{2}dx \\
&= \frac{1}{4}\int 1-\cos^2(2x)dx \\
&= \frac{1}{4}\int 1-\frac{1+\cos(4x)}{2}dx \\
&= \frac{1}{4}\int \frac{1}{2}-\frac{1}{2}\cos(4x)dx \\
&= \frac{1x}{8}-\frac{\sin(4x)}{32}+C
\end{aligned}$$

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Problem 3.

$$\int \tan(x)\sec^4 x dx$$

Solution 3. This integral would be much friendlier if there was a tangent nearby, so that $u = \tan(x)$ and $du = \sec^2(x)dx$. If I break the integrand into $\sec^2(x)(x)\sec^2(x)$ and substitute $\sec^2(x) = (1 + \tan^2(x))$, I will only have a $\sec^2(x)$ left. Let us proceed, and see what happens.

$$\begin{aligned}
\int \tan(x)\sec^4(x)dx &= \int \tan(x)(1+\tan^2(x))\sec^2(x)dx \\
&= \int (\tan(x)+\tan^3(x))\sec^2(x)dx \qquad \text{Let } u = \tan(x) \text{ then } du = \sec^2(x) \\
&= \int u + u^3 du \\
&= \frac{\tan^2(x)}{2} + \frac{\tan^4(x)}{4} + C
\end{aligned}$$

Problem 4.

$$\int \sec^3(x)dx$$

Solution 4. This integral is rather difficult in the sense that substituting a $\sec^2(x) = 1 + \tan^2(x)$ will not help because there is only one secant left, and it would not help to make a u-substitution for $u = \tan(x)$. So instead we will try integration by parts. Let $u = \sec(x)$ and $dv = \sec^(x)dx$.
Then $du = \sec(x)\tan(x)$ and $v = \tan(x)$.

$$\int \sec^3(x)dx = \sec(x)\tan(x) - \int \tan^2(x)\sec(x)dx$$

Let $u = \tan(x)$ and $dv = \tan(x)\sec(x)dx$. Then $du = \sec(x)\tan(x)$ and $v = \sec(x)$.

$$\int \sec^3(x) = \sec(x)\tan(x) - [\tan(x)\sec(x) - \int \sec^2(x)\tan(x)dx]$$

$$\int \sec^3(x) = \sec(x)\tan(x) - \tan(x)\sec(x) + \int \sec(x)\sec(x)\tan(x)dx$$

We can use u-substitution.
Let $u = \sec(x)$, then $du = \sec(x)\tan(x)dx$. Then

$$\int \sec^3(x) = \sec(x)\tan(x) - \tan(x)\sec(x) + \int udu$$

$$\int \sec^3(x) = \sec(x)\tan(x) - \tan(x)\sec(x) + \frac{\sec^2(x)}{2} + C$$

The bottom line is that you want to use your brain and creativity to solve these integrals, and most importantly, you need to know your trig identities.

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