3 pole Low Pass Butterworth IIR Filter using Bilinear Transformation

Analog $P_n(s) = \dfrac{1}{s^3+2s^2+2s+1}$; Normallized at 1Hz

$$\hat{P}_n(s) = P_n(s)\Big|_{s=\frac{s}{\omega_A}} = \frac{\omega_A^3}{s^3+2\omega_A s^2+2\omega_A^2 s+\omega_A^3}$$

$$\omega_A = \tan\left(\frac{2\pi f_c(1/f_s)}{2}\right) = \tan(\pi f_c') \quad \text{where } f_c' = \frac{f_c}{f_s}$$

f_c = cut off freq.
f_s = sample freq.

Now convert to z domain

$$H(z) = \hat{P}_n(s)\Big|_{s=\frac{z-1}{z+1}} = \frac{\omega_A^3(z+1)^3}{(z-1)^3+2\omega_A(z+1)(z-1)^2+2\omega_A^2(z+1)^2(z-1)+\omega_A^3(z+1)^3}$$

Simplify

$$\begin{array}{rcccc}
(z-1)^3 = & z^3 & -3z^2 & +3z & -1 \\
2\omega_A(z+1)(z-1)^2 = & 2\omega_A z^3 & -2\omega_A z^2 & -2\omega_A z & +2\omega_A \\
2\omega_A^2(z+1)^2(z-1) = & 2\omega_A^2 z^3 & +2\omega_A^2 z^2 & -2\omega_A^2 z & -2\omega_A^2 \\
\omega_A^3(z+1)^3 = & \omega_A^3 z^3 & +3\omega_A^3 z^2 & +3\omega^3 z & +\omega_A^3 \\
 & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\
 & A & B & C & D
\end{array}$$

$H(z) = \dfrac{Y(z)}{X(z)}$ classic input output ⇒ gain

$$\frac{Y(z)}{X(z)} = \frac{\omega_A^3(1+3z+3z^2+z^3)}{Az^3+Bz^2+Cz+D}$$

We need $\frac{Y(z)}{X(z)}$ in terms of z^{-n} for conversion to $y(n)$ samples.

$$\frac{Y(z)}{X(z)} = \frac{Y(z)\cdot z^{-3}}{X(z)\cdot z^{-3}} = \frac{\omega_A^3(z^{-3}+3z^{-2}+3z^{-1}+1)}{A+Bz^{-1}+Cz^{-2}+Dz^{-3}}$$

$$Y(z)\left[A+Bz^{-1}+Cz^{-2}+Dz^{-3}\right] = X(z)\left[\omega_A^3(z^{-3}+3z^{-2}+3z^{-1}+1)\right]$$

Now $y(n)$

$$Ay(n)+By(n-1)+Cy(n-2)+Dy(n-3) = \omega_A^3 x(n)+3\omega_A^3 x(n-1)+3\omega_A^3 x(n-2)+\omega_A^3 x(n-3)$$

$$y(n) = \frac{\omega_A^3}{A}x(n)+\frac{3\omega_A^3}{A}x(n-1)+\frac{3\omega_A^3}{A}x(n-2)+\frac{\omega_A^3}{A}x(n-3)+\frac{B}{A}y(n-1)+\frac{C}{A}y(n-2)+\frac{D}{A}y(n-3)$$

So that the filter equation is all "+" terms take the negative of B, C & D.

$$A = 1+2\omega_A+2\omega_A^2+\omega_A^3$$
$$B = 3+2\omega_A-2\omega_A^2-3\omega_A^3$$
$$C = -3+2\omega_A+2\omega_A^2-3\omega_A^3$$
$$D = 1-2\omega_A+2\omega_A^2-\omega_A^3$$