2 Pole Butterworth Low Pass

(1) Normalized Butterworth 2 Pole $P(s) = \frac{1}{s^2 + \sqrt{2}\, s + 1}$

normalized means $\omega = 2\pi f = 1$ radian/sec

(2) Scale to our -3db point $\triangleq \omega_A$ where f_c = cutoff freq

$$\omega_A = \tan\left(\frac{2\pi (f_c) T}{2}\right)$$

f_c is our desired cutoff freq

T is our systems sample period or $\frac{1}{f_s}$

f_s is our systems sample frequency

Note $f_c \times T = f_c / f_s$; f_c < Nyquest frequence

$$f_c < \tfrac{1}{2} f_s$$

(3) $$\hat{P}(s) = P(s)\Big|_{s = \frac{s}{\omega_A}} = \frac{\omega_A^2}{s^2 + \sqrt{2}\,\omega_A s + \omega_A^2}$$

$\hat{P}(s)$ is now a Butterworth at our f_c.

(4) $H(z)$ is the digital sample domain

$$H(z) = \hat{P}(s)\Big|_{s = \frac{z-1}{z+1}}$$ This is our transformation

Let $A = \omega_A^2$ and $B = \sqrt{2}\,\omega_A$

$$H(z) = \frac{A(z+1)^2}{(z-1)^2 + B(z-1)(z+1) + A(z+1)^2}$$

$$\frac{A(z+1)^2}{(z-1)^2+B(z-1)(z+1)+A(z+1)^2}$$

$$\frac{Az^2+2Az+A}{(1+B+A)z^2+(2A-2)z+(1-B+A)}$$

$$\begin{array}{ccc} z-1 & z-1 & z+1 \\ z-1 & z+1 & z+1 \\ \hline -z+1 & z-1 & z+1 \\ z^2-z & z^2-z & z^2+z \\ \hline z^2-2z+1 & z^2-1 & z^2+2z+1 \end{array}$$

$$z^2+Bz^2+Az^2-2z+2Az+1-B+A$$

$$z^2(1+B+A)+(2A-2)z^1+(1-B+A)$$

Convert to z terms

$$\frac{A+2Az^{-1}+Az^{-2}}{(1+B+A)+(2A-2)z^{-1}+(1-B+A)z^{-2}}$$

(5)

$$H(z)=\frac{Y(z)}{X(z)} \Rightarrow Y(z)[(1+B+A)+(2A-2)z^{-1}+(1-B+A)z^{-2}] = X(z)[A+2Az^{-1}+Az^{-2}]$$

Difference Equation $\Downarrow$ $y(n) : x(n)$

$$(1+B+A)y(n) = Ax(n)+2Ax(n-1)+Ax(n-2)-(2A-2)y(n-1)-(1-B+A)y(n-2)$$

$$y(n) = \underbrace{\frac{A}{1+B+A}}_{a_0}x(n)+\underbrace{\frac{2A}{(1+B+A)}}_{a_1}x(n-1)+\underbrace{\frac{A}{1+B+A}}_{a_2}x(n-2)-\underbrace{\frac{2A-2}{1+B+A}}_{b_0}y(n-1)-\underbrace{\left(\frac{1-B+A}{1+B+A}\right)}_{b_1}y(n-2)$$

$A=\omega_A^2 \quad B=\sqrt{2}\,\omega_A$

$$a_0=\frac{\omega_A^2}{1+\sqrt{2}\omega_A+\omega_A^2} \qquad b_0=\frac{2\omega_A-2}{1+\sqrt{2}\omega_A+\omega_A^2}$$

$$a_1=2a_0 \qquad b_1=\frac{1-\sqrt{2}\omega_A+\omega_A^2}{1+\sqrt{2}\omega_A+\omega_A^2}$$

$$a_2=a_0$$

For Sample Period of 18.75 ms and $f_c=3$ Hz

$$\omega_A=0.178577$$

$a_0=0.024828$	round off and set $=1.0000$ $\Rightarrow$	$a_0=0.0248$
$a_1=0.0496556$		$a_1=0.0496$
$a_2=0.024828$		$a_2=0.0248$
$b_1=1.507448$		$b_1=1.5078$
$b_2=-0.606760$		$b_2=-0.6067$
sum $=\sim 1.000000$		$=0.9999$

We need to add 0.0001 to one coef ↗

make $a_1=0.0497$

We are good!