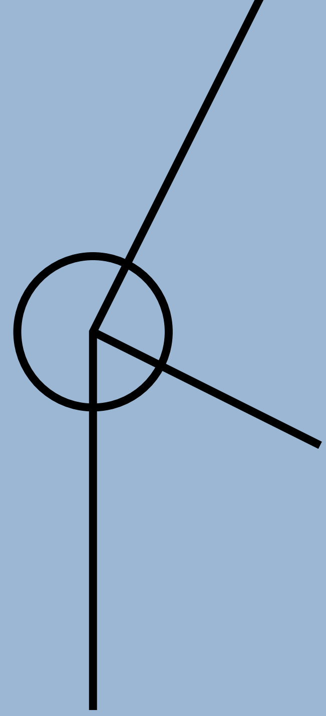
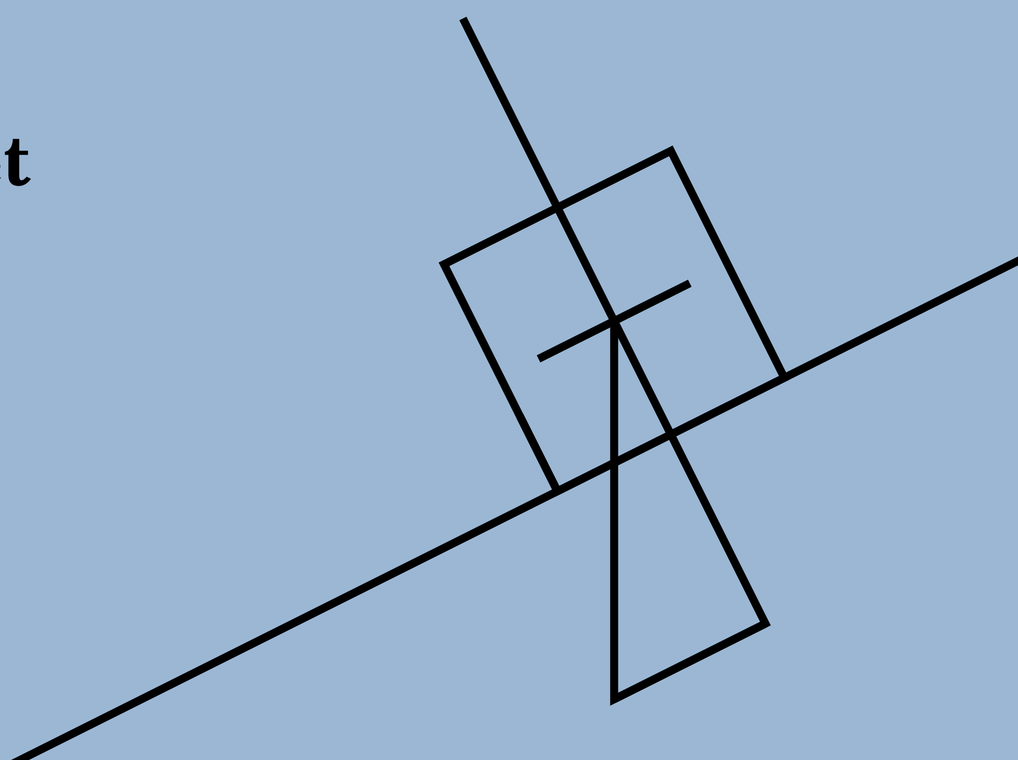


$\Sigma\vec{F}=m\vec{a}$ Physics

A Review of Concepts and Techniques for the $\Sigma\vec{F}=m\vec{a}$ Exam

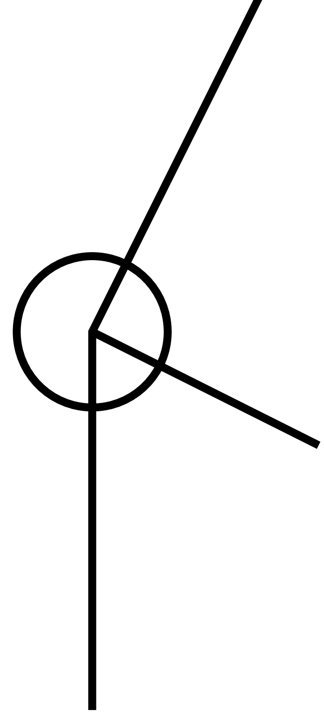
Jacob Saret



© 2018-2023 ■ Jacob Saret ■ All rights reserved.

No part of this book may be reproduced or distributed by any means for any purpose without express permission by the author.

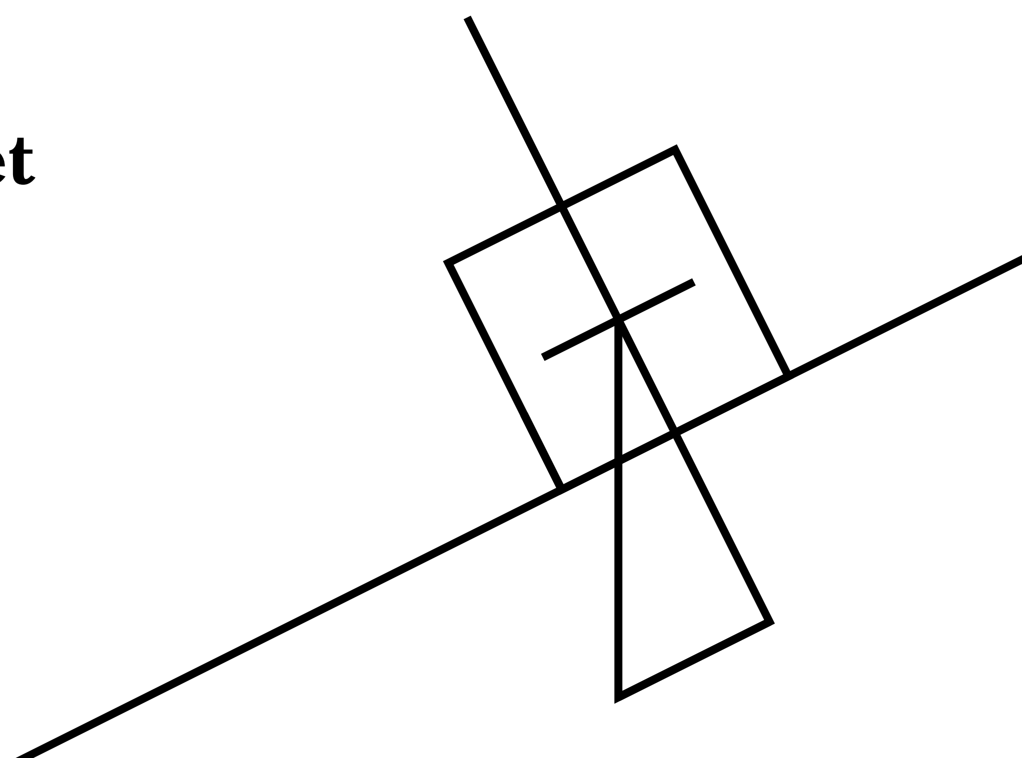
Visit `saret.co/writings` for more information.



$\Sigma\vec{F}=m\vec{a}$ Physics

A Review of Concepts and Techniques for the $\Sigma\vec{F}=m\vec{a}$ Exam

Jacob Saret



To Dr. David Brown and the San Diego Math Circle community.

Preface

This is a guide intended for students interested in preparing for the annual $\Sigma\vec{F}=m\vec{a}$ Exam, offered by the American Association of Physics Teachers (AAPT), and which serves as a qualifier for the USA Physics Olympiad (USAPhO). The content is designed as a review for a student who is already familiar with the basic physics concepts used in each Topic, and focuses on teaching approaches, techniques, and problem-solving strategies to optimize the application of their extant knowledge to questions in the $\Sigma\vec{F}=m\vec{a}$ style.

This material has been designed from scratch, tested, and refined each and every year I instructed the $\Sigma\vec{F}=m\vec{a}$ Physics class at the San Diego Math Circle. Improvements have been both from my own observations and based on student feedback and performance after every course offering.

Acknowledgements

I would like to thank the countless instructors from whom I've had the privilege of learning physics, both as a topic of study and of teaching, and the similarly countless instructors, staff, and organizers of the numerous Math Circles I have enjoyed as a member and fellow instructor.

Additionally, it is all of the students who care to learn about physics in any format that are the real reason a person like myself cares to teach. It is an honor to play even a small part in inspiring those who are on the beginning of a path where they learn not just new material, but active problem-solving and critical thinking.

San Francisco, 2023 **Jacob Saret**

Advising, Mentoring and Tutoring from Saret & Co. Education

If you are a student, or parent of a student, interested in preparing for the $\Sigma\vec{F}=m\vec{a}$ Exam, the U.S. Physics Olympiad, or otherwise looking for support in your studies of physics and mathematics at any level, you may be interested in the services available from Saret & Co. Education.

Saret & Co. Education is led by Jacob Saret, the author of this guide, creator and long-time instructor of the $\Sigma\vec{F}=m\vec{a}$ Physics course at the San Diego Math Circle. He is a seasoned advisor, mentor and tutor specializing in gifted and non-traditional education. Jacob works directly with learners of all ages in these capacities, and also personally oversees expert contract tutors serving a variety of student needs with a range of teaching styles and approaches.

Jacob has extensive personal experience through over a decade of non-traditional education from leaving elementary school at age 7 to beginning college at age 10, university at 15, and graduate school at 18. He has worked with countless private tutors, participated in numerous summer camps, many gifted-specific programs, and more. Over the course of his education, he studied at three California Community Colleges and three University of California campuses. In addition to his traditional expertise, he spent several years as a student at the Stanford Online High School, giving him the unique ability to offer fully-remote tutoring that works for each individual student, without compromising the quality of instruction, and to guide all of our tutors on how to do the same.

Jacob has been a private tutor in physics and mathematics since 2015, and a teaching assistant in the physics departments at UCLA and UCSD from 2017 to 2021. In total, he has worked directly with over 500 students. Furthermore, all tutors at Saret & Co. Education have a range of expertises and experiences working with learners of diverse backgrounds, learning styles, and needs varying from advancement to remedial to enrichment.

If you are interested in learning more, or would like to work with Jacob, or any member of the excellent Saret & Co. Team, please reach out to Jacob through the Saret & Co. Education website, `saret.co`.

More information can be found on the webpages linked below.

General Information	`saret.co`
Advising and Mentoring	`saret.co/advising-and-mentoring`
Tutoring	`saret.co/tutoring`
The Saret & Co. Team	`saret.co/tutors`
About Jacob	`saret.co/about`
Testimonials	`saret.co/testimonials`
Contact	`saret.co/contact`

We look forward to hearing how we can be a part of your educational journey.

Sincerely,
— Jacob

Jacob Saret
Director • Gifted Education Advisor & Mentor • Principal Tutor
Saret & Co. Education

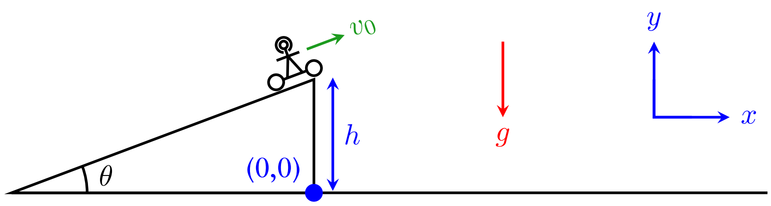
Table of Contents

Topic 1 ■ Kinematics

Kinematics is the study of objects in motion, without requiring the consideration of forces. A plurality of $\Sigma\vec{F}=m\vec{a}$ Exam questions rely on the application of kinematics concepts, even if the focus of the question is on more advanced topics.

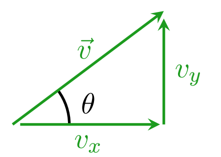
1·1 ■ Algebraic Kinematics

Question. Consider a stuntwoman jumping a motorcycle off a ramp of height h and incline angle θ, with launch velocity v_0. How far away from the ramp will the motorcycle land?



Solution. First, recall that x and y motion can be treated independently. So, we will begin with a look at her vertical motion to see how long she's airborne.

First, we split v_0 into x and y components, as seen at right,



$$\begin{aligned} v_{0y} &= v_0 \sin\theta \\ v_{0x} &= v_0 \cos\theta \end{aligned}$$

A technique to recall which term takes the sine and which takes the cosine is presented in **Topic 9**. Given this, her vertical trajectory is described by

$$y_f = y_0 + v_{0y}t + \left(-\frac{1}{2}g\right)t^2 \implies 0 = h + v_{0y}t + \left(-\frac{1}{2}g\right)t^2.$$

This is a quadratic equation with two solutions. Only one of the solutions is positive, so this is the only sensible answer. The negative solution must be discarded, as our model does not consider $t < 0$ and any results in that domain are consequently meaningless. So,

$$t = \frac{v_{0y} \pm \sqrt{{v_{0y}}^2 - 4h(-g/2)}}{2(-g/2)} \implies t = -\frac{v_0\sin\theta - \sqrt{{v_0}^2\sin^2\theta + 2hg}}{g}.$$

We can substitute this into the equation for her motion in the x direction, $x_f = x_0 + v_{0x}$, to find

$$x_f = -v_0\cos\theta\left(\frac{v_0\sin\theta - \sqrt{{v_0}^2\sin^2\theta + 2hg}}{g}\right)$$

which is now ready to have numerical values substituted, had it been requested by the question.

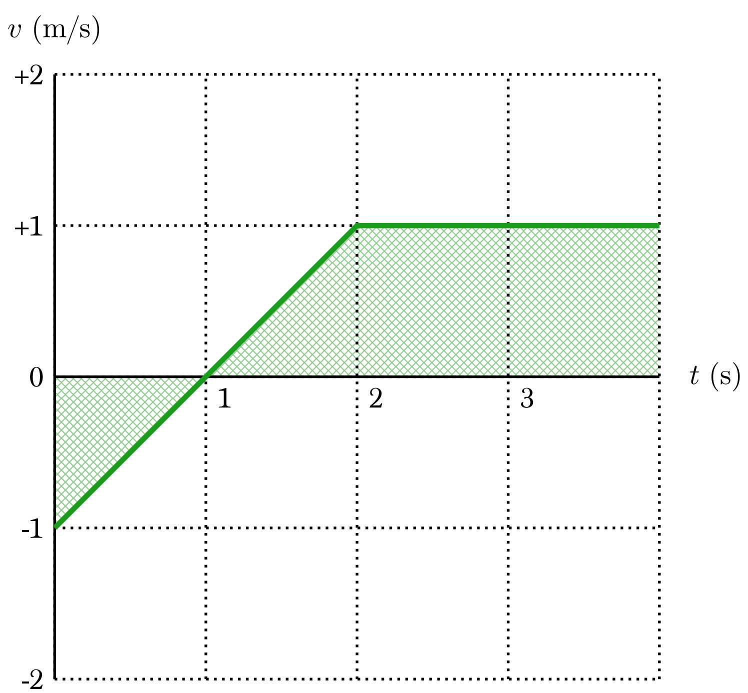
1·2 ■ Graphical Kinematics

In the several plot-based kinematics questions on each $\Sigma\vec{F}$=m$\vec{a}$ Exam, the horizontal axis usually denotes time. When this is the case, velocity is the slope (time derivative) of position, and acceleration is the slope of velocity. Correspondingly, position is the area under (integral over time of) velocity, which itself is the area under acceleration.

$$v = \frac{dx}{dt} \iff \int v dt = x$$
$$a = \frac{dv}{dt} \iff \int a dt = v$$

It is always paramount to check that the horizontal axis of a plot shows time, before applying these techniques to the plot in question.

Question. In the velocity-time graph below, when the object under observation begins its movement at $t = 0$ s, its position is $x_0 = -1.5$ m. When does it pass the origin, and what is its acceleration at that time?



Solution. Summing area of the green region between the beginning of the motion and time t tells us the position at that time. At $t = 1$ s, the area is 0.5 below the axis, so

$$x(t = 1 \text{ s}) = x_0 - 0.5 \text{ m} = -2 \text{ m}.$$

By this logic, the area totals 1.5 m at $t = 3.5$ s. So, this is when the object passes through $x = 0$ m.

The slope of the velocity-time graph tells us the acceleration of the object. For $0 \text{ s} < t < 2 \text{ s}$, the object's velocity is increasing by 1 m/s per second, then remains constant for $t > 2$ s. So,

$$a(x = 0 \text{ m}) = a(t = 3.5 \text{ s}) = 0 \text{ m/s}^2.$$

Topic 2 ■ Forces and Newton's Laws

Considering that Newton's 2nd Law is verbatim the name of the $\Sigma\vec{F}=m\vec{a}$ Exam, being familiar with Newton's Laws and comfortable applying them to a variety of questions might be the single most important skill to have down pat come test day.

2·1 ■ Newton's Laws

Newton's 1st Law. Objects in motion stay in motion [at constant velocity], and objects at rest stay at rest, unless subject to a [net external] force.

Newton's 2nd Law. The namesake of this exam. The net force on an object is equal in magnitude and direction to the mass of the object times its acceleration. In equation form,

$$\vec{F} = m\vec{a}$$
$$\text{net force on object} = \text{mass of object} \cdot \text{acceleration}$$

Newton's 3rd Law. Every action [(force)] has an equal and opposite reaction [(force)].

Note that these laws also apply to entire systems, when the objects compromising that system are static in the reference frame of the system.

These laws will be applied to countless questions, and when you need one you may need them all. Many questions will involve force balance or solving for net accelerations, with a subsequent application of the kinematics techniques discussed in **Topic 1**.

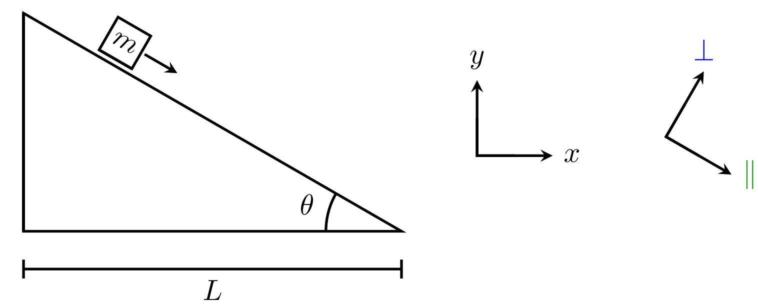
Question. A block of mass $m_1 = 5$ kg is resting upon a crate of mass $m_2 = 10$ kg, which is sitting on a surface with coefficient of kinetic friction $\mu = 0.5$. What magnitude of force needs to be applied to the crate in order to slide them together at constant velocity?

Solution. Because the box and the crate are not moving relative to each other, we can treat this question as one system, with mass $m = 15$ kg. As such, the force of friction is $F_\mu \leqslant \mu m g = 75$ N.

If we want the crate-box system to move with constant velocity, Newton's 2nd Law requires there be no net force, so the force with which we push the crate must be equal in magnitude to the maximum force of kinetic friction. Thus, $F_{\text{push}} = 75$ N.

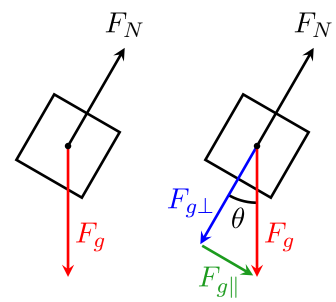
2.2 ■ Inclined Planes

Question. How long does it take a box with mass $m = 25$ kg to slide down a frictionless ramp sloping at $\theta = 30°$, with a base $L = 10$ m long?



Solution. Let's start by making a free body diagram, and and decompose F_g into $\perp$ and $\parallel$ components. Defining and switching to your own new coordinate system, such as converting (x, y) to $(\perp, \parallel)$, is a very useful technique for many ΣF⃗=ma⃗ Exam questions, and beyond.

The box remains on the surface of the plane, so $\vec{F}_{\text{net}\perp} = 0$. This reveals two vector equations for us to solve, which can be rewritten as scalar equations.



$$\vec{F}_{\text{net}\perp} = 0 = \vec{F}_N + \vec{F}_{g\perp} \implies F_N = -F_{g\perp}$$
$$\vec{F}_g = \vec{F}_{g\perp} + \vec{F}_{g\parallel} \implies F_{g\perp} = F_g \cos\theta$$

With an application of the Pythagorean Theorem, we see

$$F_g{}^2 = F_g{}^2 \cos^2\theta + F_{g\parallel}^2 \implies F_{g\parallel} = F_g \sin\theta.$$

Now we're ready to substitute some numbers, namely $\theta = 30°$ and $m = 25$ kg,

$$a_\parallel = \frac{F_{\text{net}\parallel}}{m} = \frac{F_{g\parallel}}{m} = \frac{F_g}{m}\sin\theta = \frac{mg}{m}\sin\theta = g\sin\theta = 5\ \text{m/s}^2.$$

Note that only the length of the base of the ramp is given. The length of the surface of the ramp is the length of the hypotenuse of the triangle, which is $(10\ \text{m})/(\cos 30°) \approx 11.5$ m. Now, we can conclude our solution with a kinematics equation in the $\parallel$ direction.

$$x_\parallel = x_{\parallel,0} + \frac{1}{2}a_\parallel t^2 \implies 11.5 = 0 + \frac{1}{2}\cdot 5 \cdot t^2 \implies t = 2.2\ \text{s}$$

Remark. The mass of the box did not affect the motion in any way relevant to answering this question. This is a result of the combination of linear scaling of the force of gravity with mass, and the inverse scaling of acceleration with mass. Though unique to gravity, this motif appears with sufficient frequency on the ΣF⃗=ma⃗ Exam to deserve your attention when it emerges during your studies.

Topic 3 ■ Simple Harmonic Oscillators

Ever-present on $\Sigma\vec{F}$=m$\vec{a}$ Exams, simple harmonic oscillators, often referred to as SHOs, are systems acted upon by one and only one precisely linear restoring force. That is, the restoring force acting on the system has a form $F(x) = -kx$, where x is the characteristic displacement of the system and k is the force constant. In this case, the equation of motion could be written as

$$\begin{aligned} F &= -kx \\ ma &= -kx \\ m\ddot{x} &= -kx \end{aligned}$$

This differential equation, which you need not know how to solve for the $\Sigma\vec{F}$=m$\vec{a}$ Exam, has solutions of the form

$$x = A\cos(\omega t + \phi)$$

with amplitude A, given by the maximum displacement from equilibrium, phase shift ϕ, and natural oscillation frequency

$$\omega = \sqrt{\frac{k}{m}}.$$

The energy stored in this system is given by an application of the work-energy theorem to be

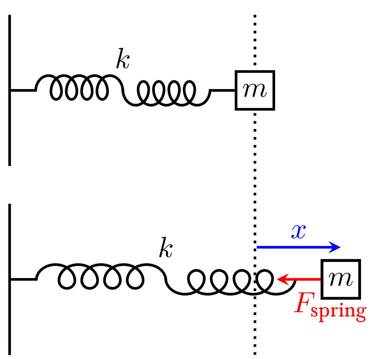
$$E_{\text{osc}} = \frac{1}{2}kA^2.$$

These results are likely familiar to you as modeling the simple harmonic motion of a mass on a spring. However, note that this derivation does not describe the progenitor of the restoring force, only its form, and thus applies to many systems, as shown in this Topic.

3·1 ■ Spring Oscillators

Question. Consider a simple system of one mass on a single spring. What can we say about the forces and energy in this system? What is its natural oscillation frequency?

Solution. Hooke's Law tells us that the restoring force from a spring stretched a distance $\vec{x}$ from equilibrium is $\vec{F} = -k\vec{x}$. This tells us that the natural frequency of oscillation of this system is $\omega = \sqrt{k/m}$.



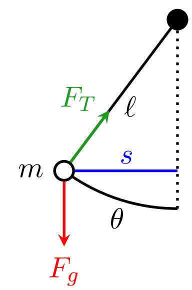
All questions on the $\Sigma\vec{F}$=m$\vec{a}$ Exam concerning springs with more intricate setups than this are focused on determining a new or effective spring constant. They may require an application of this spring constant to determine a new natural oscillation frequency, but the spring constant itself is the key. This is discussed at length and in several geometries in **Topic 4**.

Notice that all quantities in this question which depend on position are in fact dependent on displacement. So, if a spring system were subject to a constant external force, we simply adjust the equilibrium position so the additional spring force balances the external forces. This is explored in **Topic 4·4**.

3·2 ■ Pendulums and the Small Angle Approximation

On the $\Sigma\vec{F}=m\vec{a}$ Exam, we assume rotational oscillations take place over a small angle θ, unless we are explicitly told otherwise. This allows us to make the crucial simplification $\sin\theta \approx \theta$.

Question. What is the natural oscillation frequency of a simple pendulum which consists of a bob of mass m on a massless string of length ℓ?



Solution. Trigonometry says that the component of F_T along the path of the pendulum is $F_{T\theta} = F_T \sin\theta \approx F_T\theta$, with $F_T \approx -F_g$. So,

$$F_{T\theta} \approx -F_g\theta = -F_g\frac{s}{\ell}.$$

In the small angle approximation, true path of the pendulum is taken to be equivalent to the horizontal path s. Since $F_{T\theta}$ is along this path, we can see this system is subject to is a restoring force $F_{T\theta}(s) \propto -s$.

As derived at the beginning of this Topic, any system with a restoring force characterized by a force constant k has a natural oscillation frequency of $\omega = \sqrt{k/m}$. Here, $k = mg/\ell$, so the natural oscillation frequency of our simple pendulum is

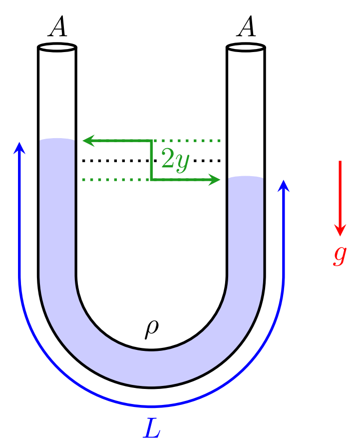
$$\omega = \sqrt{\frac{g}{\ell}}.$$

Physical pendulums, those with mass distributions more complicated than a single mass at the end of a string, are discussed in **Topic 5·3**.

3·3 ■ Other Oscillators

All other oscillating systems on the $\Sigma\vec{F}=m\vec{a}$ Exam will be dealt with in the same manner as those shown throughout this Topic. This is illustrated through a question about a fluid-based oscillator.

Question. Consider a U-tube manometer of constant cross-sectional area A filled with water with density ρ, as shown at right. When at rest, the total volume of the water is $V = LA$. If the water is perturbed by a distance y, what is the frequency of oscillation of the water?



Solution. The key to answering this question is finding the restoring force on the water in the manometer. This force will be due to the weight of $2yA$ of displaced water, acting on a total mass of ρLA of water. Thus, we can say

$$F = mg = 2\rho yAg = -ky \implies \omega = \sqrt{\frac{2\rho Ag}{\rho LA}} = \sqrt{\frac{2g}{L}}.$$

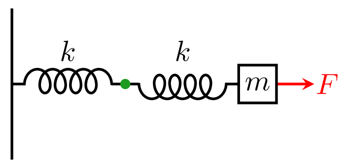
Notice this is only dependent on a couple of physical parameters, such as the total amount of water L and the gravitational acceleration g, but not the density of water ρ, the magnitude of the displacement y, nor the geometry of the manometer, which only appears in the question as A.

Topic 4 ■ Springs

Many questions on the $\Sigma\vec{F}=m\vec{a}$ Exam deal with oscillation of spring systems, with geometries beyond the simple one outlined in **Topic 3·1**. Questions on these complex spring constructions focus on determining a new or effective spring constant, which is sometimes applied to determine the natural oscillation frequency of the system.

4·1 ■ Springs in Series

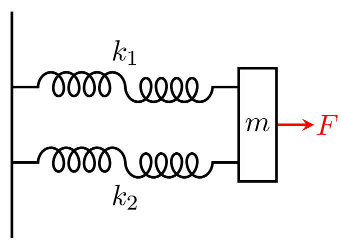
Question. Consider two identical springs joined end-to-end, or in series. If we exert a force F on the mass m at the end of this system, by how much will it be displaced?



Solution. First, for any displacement x the mass experiences, we observe from the symmetry of the system that the green point will be displaced by $x/2$. That point also has F exerted on it, by both the spring on its left and its right, as all of these are in equilibrium once the system has reached its new position. So, $x/2 = F/k$, and thus $x = 2F/k$. Given the definition $k \equiv F/x$, we see that $k_{\text{series}} = k/2$.

Exercise. Rework this system when the two springs have different spring constants, $k_1 \neq k_2$.
Hint. The equilibrium condition is the same.

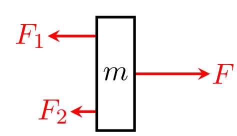
4·2 ■ Springs in Parallel



Question. Examine this system two springs with equal rest lengths, but different spring constants, joined in parallel. If we exert a force F on the mass m, how far will it be displaced?

Solution. Since both springs are extended by the same x, we can write a system of two equations,

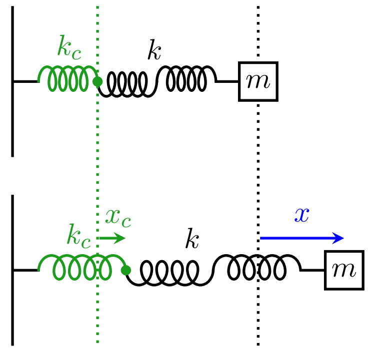
$$x = \frac{F_1}{k_1} = \frac{F_2}{k_2} \quad \text{and} \quad F = F_1 + F_2.$$



Solving this for F, we see that $k_1 x + k_2 x = F$, and $x = F/(k_1 + k_2)$. The definition of the spring constant implies that $k_{\text{parallel}} = k_1 + k_2$.

4·3 ■ Springs in Pieces

Question. Consider a uniform spring with spring constant k and unstretched length L, which has been cut into two pieces. One piece has an unstretched length $\frac{1}{N}L$, where N is a dimensionless number. What is the spring constant of this new piece?



This scenario is illustrated at right with $N = 3$. The green point represents the point where the spring was cut, and the piece of the spring in question is the green segment.

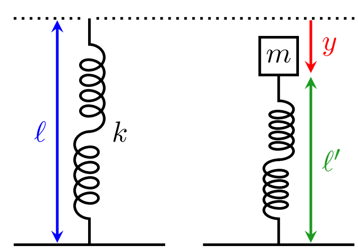
Solution. With a force F applied to the end of the original spring, we know both pieces of the spring have the same F acting on each, since the green point is not accelerating. Therefore, the forces acting on the green segment are of magnitude $F_c = F$. Geometrically, we see that $x_c = x/N$. Recalling that the spring constant k is defined as $k \equiv F/x$, we write

$$k_c = \frac{F_c}{x_c} = \frac{NF}{x} = Nk.$$

Remark. The spring constant k is not solely a material property of the spring, but also depends on its construction. Otherwise, this result would be quite inexplicable.

4·4 ■ Springs in Gravity

Question. Imagine a massless vertical spring with equilibrium length ℓ. A mass m is placed upon this spring. What is the equilibrium position of this system, and what is its natural oscillation frequency?



Solution. Equilibrium is achieved when the force of gravity on the mass is balanced by the spring force. In equation form,

$$mg = ky \implies y = \frac{mg}{k} \implies \ell' = \ell - \frac{mg}{k}.$$

Now, let us consider the oscillation frequency of this system. Recall that the derivation of the natural oscillation frequency presented in **Topic 3** was not dependent on the absolute rest position of the system, but rather its displacement from that rest position.

This condition remains satisfied if the system is modified, as long as the restoring force is still linear in displacement. Fortunately, even from an adjusted equilibrium length ℓ', the force resulting from an additional displacement d is still

$$F(d) = -kd$$

and natural oscillation frequency of this system is the familiar $\omega = \sqrt{k/m}$.

Remark. This simple adjustment is does not apply to calculating the energy stored in the spring. In the above question, where the spring is compressed a distance y in its new equilibrium position, the energy the spring will store under additional compression d is

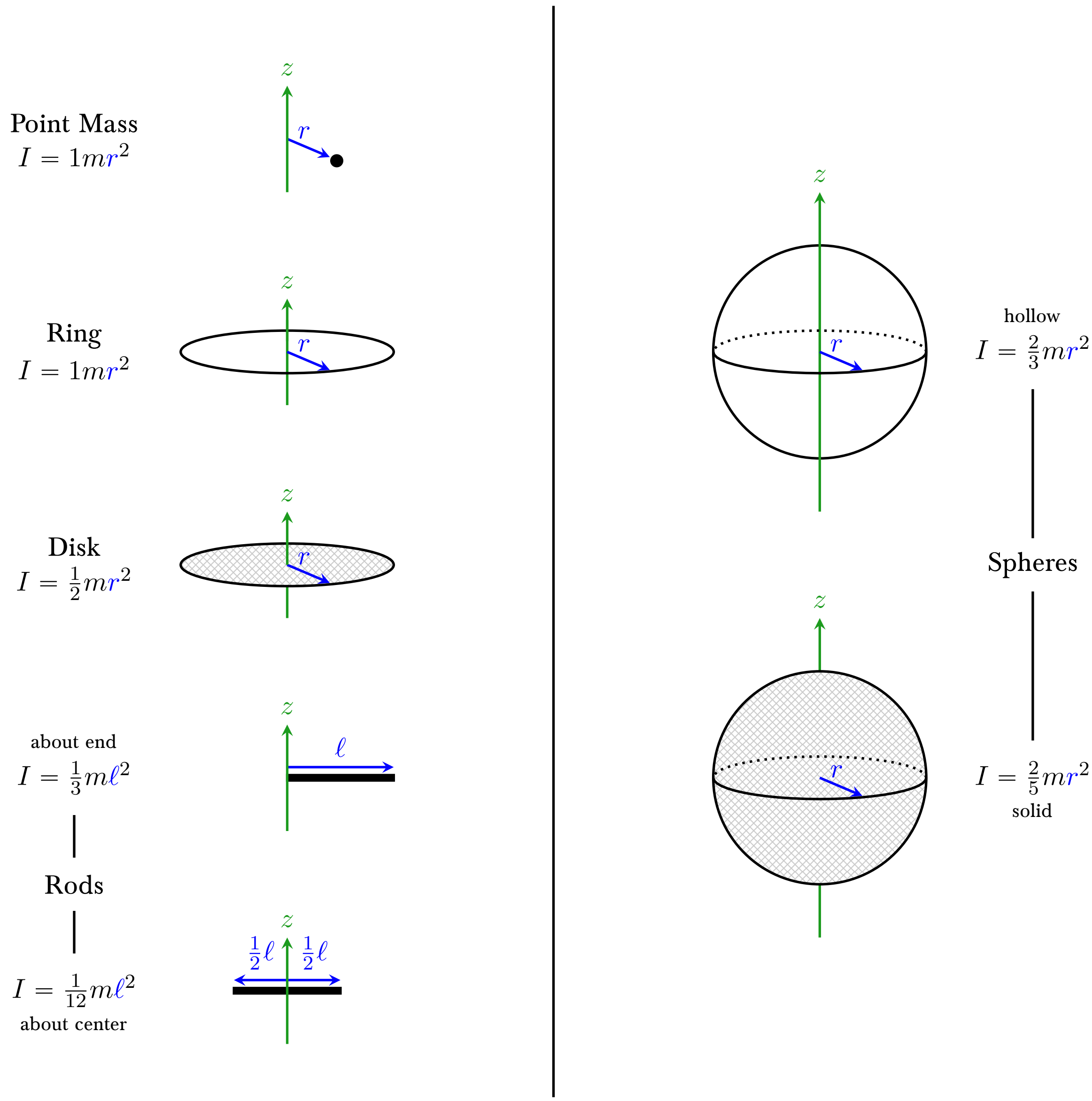
$$\Delta E = \frac{1}{2}k\,(y+d)^2 - \frac{1}{2}ky^2 = \frac{1}{2}kd^2 + kyd.$$

Observe the additional term kyd, which is linear in d and, notably, does depend on whether y and d are parallel or antiparallel.

Topic 5 ■ Moments of Inertia

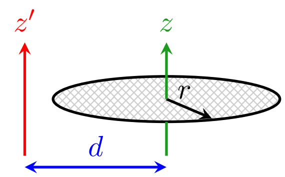
Many questions on the $\Sigma\vec{F}=m\vec{a}$ Exam require you to know the moments of inertia of so-called common objects. The seven shapes shown in **Topic 5·1**, together with the parallel axis theorem presented in **Topic 5·2**, should suffice. At points in this guide, we will use the moment of inertia coefficient κ to represent objects possessing a moment of inertia $I = \kappa m r^2$, where r is the length scale of the object.

5·1 ■ Moments of Common Objects



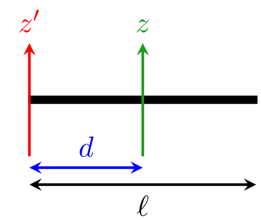
Remark. Extruding symmetric shapes in z does not change the value of κ. Moments depend solely on distribution of mass and distance of each element, or chunk of mass, from the axis of rotation. **Corollary**. If an object is stretched in z, and its mass remains constant, the value of I will not change.

5·2 ■ Parallel Axis Theorem



The parallel axis theorem states that the moment of inertia of an object about an axis of rotation that does not pass through its center of mass is the moment of inertia of that object, plus the total mass times the square of the distance between these two axes. In mathematical form,

$$I_z' = I_z + md^2.$$



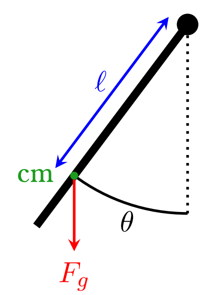
Perhaps this is most clearly demonstrated by considering the moments of inertia of a rod of length ℓ about its center and about its end.

$$I_z' = I_z + md^2 = \frac{1}{12}m\ell^2 + m\left(\frac{\ell}{2}\right)^2 = \frac{1}{3}m\ell^2$$

5·3 ■ Physical Pendulums

Some pendulums on the $\Sigma\vec{F}=m\vec{a}$ Exam are not the simple kind, where there is a mass m at the end of a rope or rod of length ℓ. Instead, they are physical pendulums, characterized by some mass distribution possessing a moment of inertia I, about whichever axis of rotation they are pivoted upon.

Note that this moment of inertia is distinct from the moment of inertia about the center of mass of an object and calculating it usually requires an application of the parallel axis theorem.



Question. What is the period of oscillation of a nonuniform rod of mass m which is pivoted at one end a distance ℓ from its center of mass? The rod has moment of inertia I about the axis of rotation passing through this pivot point.

Solution. We can treat the whole rod as a point mass m located at the center of mass of the rod. Using the rotational equivalent of Newton's 2nd Law, and a small angle approximation,

$$\tau = I\alpha = -F_g\ell\sin\theta \approx -mg\ell\theta.$$

This result is analogous to what we saw for a simple pendulum in **Topic 3·2**. We can follow the derivation exhibited there to find

$$\omega = \sqrt{\frac{mg\ell}{I}}.$$

Exercise. Apply this process to determine the natural oscillation frequency for a uniform disk pivoted a distance $R/2$ from its center.

Topic 6 ■ Rotational Dynamics

Numerous questions on the $\Sigma\vec{F}=m\vec{a}$ Exam deal with rotational dynamics, often in concert with translational motion. Notably, every motion can be described as combination of translation and rotation.

6·1 ■ Parallels of Rotation and Translation

Let's begin by considering rotational motions alone before combining them. Fortunately, rotational dynamics are strongly analogous to linear systems.

Rotation	Relations	Translation
$\vec{\tau}$	$\vec{\tau} = \vec{r} \times \vec{F}$ $\tau = r_\perp F = rF_\perp$	$\vec{F}$
$\vec{\alpha}$	$\vec{\alpha} = \frac{\vec{a}}{r}$ | $\vec{a} = r\vec{\alpha}$	$\vec{a}$
$\vec{\omega}$	$\vec{\omega} = \frac{\vec{v}}{r}$ | $\vec{v} = r\vec{\omega}$	$\vec{v}$
$\vec{\theta}$	$\vec{\theta} = \frac{\vec{x}}{r}$ | $\vec{x} = r\vec{\theta}$	$\vec{x}$
I	$I \propto mr^2$ See **Topic 5**	m
K	$K = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$	K

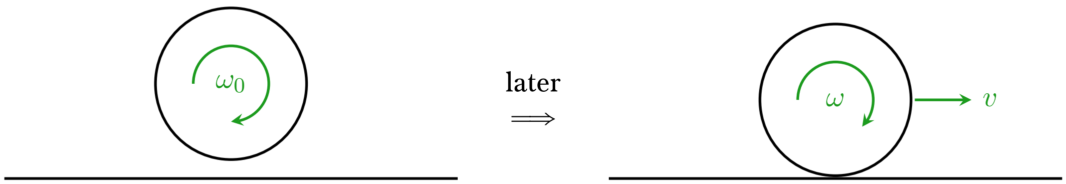
These relations enable us to convert our favorite equations and relations from basic kinematics and dynamics to rotational equivalents. For example,

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \implies \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2.$$

6·2 ■ Rotational Energy

This question pertains to an experiment performed to find the moment of inertia of an object.

Question. A ball of mass m and radius R is gently dropped on a surface, while spinning with angular velocity ω_0. Later, as it is rolling without slipping, it has a translational velocity v. No energy is lost to the surface during this motion. Determine the moment of inertia coefficient κ for this ball.



Solution. Being told no energy is lost to the surface is a strong indicator we should use conservation of energy to answer this question. We can write the initial rotational energy of the sphere,

$$E_0 = \frac{1}{2}\kappa m R^2 {\omega_0}^2.$$

When it's rolling without slipping, $v = \omega R$. So,

$$E = \frac{1}{2}mv^2 + \frac{1}{2}\kappa m R^2\omega^2 = \frac{1}{2}\left(mR^2\omega^2 + \kappa m R^2\omega^2\right) = \frac{1}{2}\left(1+\kappa\right)mR^2\omega^2.$$

As we said at the beginning of this solution, energy is conserved. So, we can equate E_0 and E to see

$$\frac{1}{2}\left(1+\kappa\right)mR^2\omega^2 = \frac{1}{2}\kappa m R^2 {\omega_0}^2 \implies \left(1+\kappa\right)\omega^2 = \kappa {\omega_0}^2.$$

Solving this for κ reveals

$$\kappa = \frac{\omega^2}{{\omega_0}^2 - \omega^2} = \frac{v^2}{{\omega_0}^2R^2 - v^2}.$$

If numerical values were given for ω_0, v, and R, we could now calculate κ.

While discussing this question, a student asked "What would happen when the object stops slipping? Why doesn't it keep accelerating the same way it has been and start slipping the other way?"

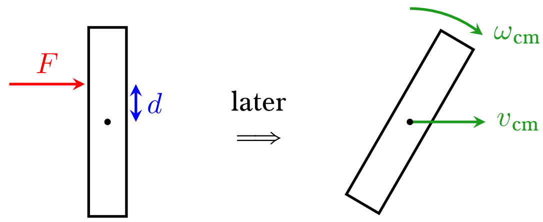
The subtlety behind this remark arises from the difference between static and kinetic friction, even when $\mu_s = \mu_k = \mu$. When the ball is both rolling and slipping, it is subject to kinetic friction, which is given by $F_k = \mu_k N$.

However, once the ball reaches the exact speed where $v = \omega R$, the contact point between the ball and surface is no longer slipping. Consequently, the frictional force enters the static regime, where $F_s \leqslant \mu_s N$. As such, the force of static friction is able to provide precisely the amount of force and torque required to keep the ball rolling without slipping.

6·3 ■ Rotational Impulse

This question concerns the delivery of a fixed impulse to a rod, leading to simultaneous translational and rotational motion.

Question. A force F is applied to a uniform rod of mass m and length ℓ at a distance d from its center of mass over a short interval Δt. Describe the subsequent motion of this rod.



Solution. First, let's approach the translational component of the motion. Newton's 2nd Law says

$$F = ma = m\frac{dv}{dt} = \frac{dp}{dt} = \frac{\Delta p}{\Delta t} \text{ so that } F\Delta t = J = \Delta p$$

where J is an impulse. From here, we can write

$$J = F\Delta t = ma\Delta t = mv_{\text{cm}} \implies v_{\text{cm}} = \frac{F\Delta t}{m}$$

which, perhaps surprisingly, is independent of d. Now, let's discuss the rotational component. By the same logic,

$$J_\tau = F\Delta t d = I\alpha\Delta t = I\omega_{\text{cm}} = \frac{1}{12}m\ell^2\omega_{\text{cm}} \implies \omega_{\text{cm}} = 12\frac{Jd}{m\ell^2}.$$

Upon seeing the solution to this question, a student asked "Does d affect the energy imparted on the rod?" Well, the energy imparted on the rod is equal to the work done by this force on the rod.

$$\Delta E = F\Delta y = J\frac{\Delta y}{\Delta t}.$$

If Δt is small, $\Delta y \approx d\Delta\theta + \Delta x$, which means

$$\alpha = \frac{Jd}{I\Delta t} \text{ and } a = \frac{J}{m\Delta t}.$$

Rotational kinematics says

$$\Delta x = \frac{1}{2}a\Delta t^2 = \frac{J\Delta t}{2m} \text{ and } \Delta\theta = \frac{1}{2}\alpha\Delta t^2 = \frac{Jd\Delta t}{2I} = 12\frac{d}{\ell^2}\frac{J\Delta t}{2m}$$

from which it follows that

$$\Delta y = \frac{J\Delta t}{2m}\left(1 + 12\frac{d^2}{\ell^2}\right) \implies \Delta E = \frac{J^2}{2m}\left(1 + 12\frac{d^2}{\ell^2}\right)$$

which sensibly depends on d.

6·4 ■ The Coriolis Effect

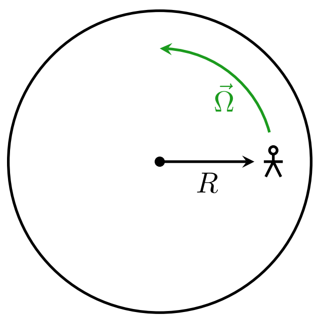
The Coriolis effect is the presence of a fictitious Coriolis force which exists in rotating reference frames, much like the centrifugal force. Most $\Sigma\vec{\text{F}}$=m$\vec{\text{a}}$ Exams have questions pertaining to the Coriolis effect, though many are graphical in nature rather than requiring explicit calculation. Graphical questions almost invariably require use of the right hand rule to determine the direction of the Coriolis force.

The Coriolis force acting on an object of mass m is given by

$$\vec{F}_{\text{coriolis}} = -2m\vec{\Omega} \times \vec{v}$$

where $\vec{\Omega}$ is the rotational velocity of the frame and $\vec{v}$ is the object's velocity relative to the frame. Make sure to account for the negative sign when performing your right hand rule gestures.

Question. Consider a child on a spinning merry-go-round, holding a coin. What does he observe if he **(a)** drops the coin, **(b)** throws the coin towards the center of the merry-go-round, or **(c)** throws the coin vertically upwards?



Solution. In this question, $\vec{\Omega}$ is pointing out of the page, as per the right hand rule.

(a) If he drops the coin, $\vec{v}_{\text{coin}}$ is parallel to $\vec{\Omega}$, so their cross product is zero and there is no Coriolis force. The coin will fall at his feet.

(b) If he throws the coin towards the center of the merry-go-round, we have

$$\vec{F}_{\text{coriolis}} \sim \vec{\Omega} \times \vec{v} \;\; \text{which has the direction} \;\; +\hat{z} \times -\hat{r} = +\hat{\theta}$$

telling us the coin gets ahead of the child, as it is being accelerated in the direction of his motion.

(c) If he throws the coin up above his head, by the same argument we saw in **(a)**, he could catch it right in his hands.

Topic 7 ■ Orbits

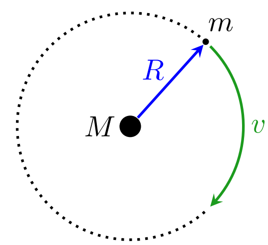
Orbit-based questions are fairly common on the ΣF⃗=ma⃗ Exam. The simplest kind, circular orbits, are characterized by the straightforward requirement that the radial force matches the force corresponding to the centripetal acceleration.

Remark. A centripetal force, while responsible for a centripetal acceleration, is not physically caused by any interaction, but is instead either a net force, or a result of another force. This could be any force on the orbiting object, but is most commonly gravity or tension. There is no "centripetal force."

7·1 ■ Circular Orbits

Question. An object of mass m orbits a much more massive, stationary body of mass M, in a perfectly circular path with a radius R, subject only to gravity. What is the period of this orbit?

Solution. As stated in the beginning of this Topic, we will begin by equating the radial force to the centripetal acceleration.



$$F_{\text{centripetal}} = \frac{mv^2}{R} = \frac{GMm}{R^2} = F_{\text{gravitational}}.$$

So, for a given R,

$$v = \sqrt{\frac{GM}{R}}.$$

At constant v and R, each orbit traverses one circumference, a distance of $2\pi R$, so

$$T = 2\pi\frac{R}{v} = 2\pi\frac{R}{\sqrt{\frac{GM}{R}}} = 2\pi\sqrt{\frac{R^3}{GM}}.$$

7·2 ■ Non-Gravitational Orbits

This result applies to non-gravitational orbits. Regardless of the force on an orbiter, orbits are stable when $F \propto R^{-2}$, and all have the same form. We will take $F = fR^{-2}$, without loss of generality. Then,

$$fR^{-2} = F = \frac{mv^2}{R} \implies v = \sqrt{\frac{f}{mR}}.$$

For orbits governed by gravity, we get a familiar result,

$$f = GMm \implies v = \sqrt{\frac{GM}{R}}.$$

The proof that orbits are stable when $F \propto R^{-2}$ is interesting, but beyond the scope of this guide.

7·3 ■ Kepler's Laws

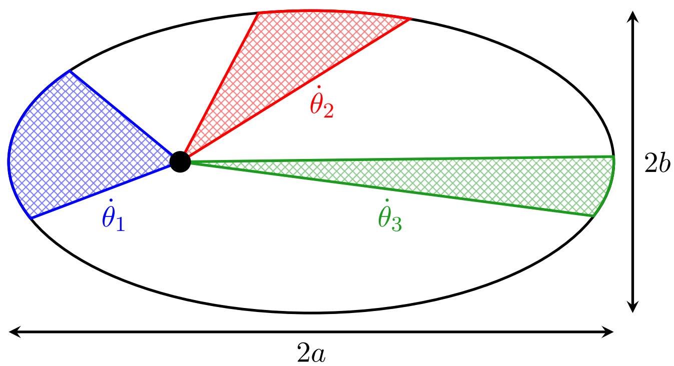
Kepler's 1st Law. All bound orbits are elliptical with the orbited body at a focus. Notice that circular orbits are elliptical with the two foci at the same point in space. The various geometries of orbits, both bound and unbound, are discussed further in **Topic 7·4**.

Kepler's 2nd Law. A line joining orbiter and orbited sweeps equal areas in equal times.

$$T \cdot \frac{r^2}{2} \cdot \frac{d\theta}{dt} = \pi ab$$

$$t_1 = t_2 = t_3$$

$$\dot{\theta}_1 > \dot{\theta}_2 > \dot{\theta}_3$$



There is a subtle but important distinction between $\dot{\theta} = d\theta/dt$ and ω, due to the speed of the orbiting object changing continuously throughout any non-circular orbit. If you are unfamiliar with this notation, don't fret. You will likely not encounter it again until advanced mechanics in your sophomore or junior year of college physics.

Kepler's 3rd Law. The square of the period T of an orbit is proportional to the cube of the radius R of the same closed orbit, as derived in **Topic 7·1**. In equation form,

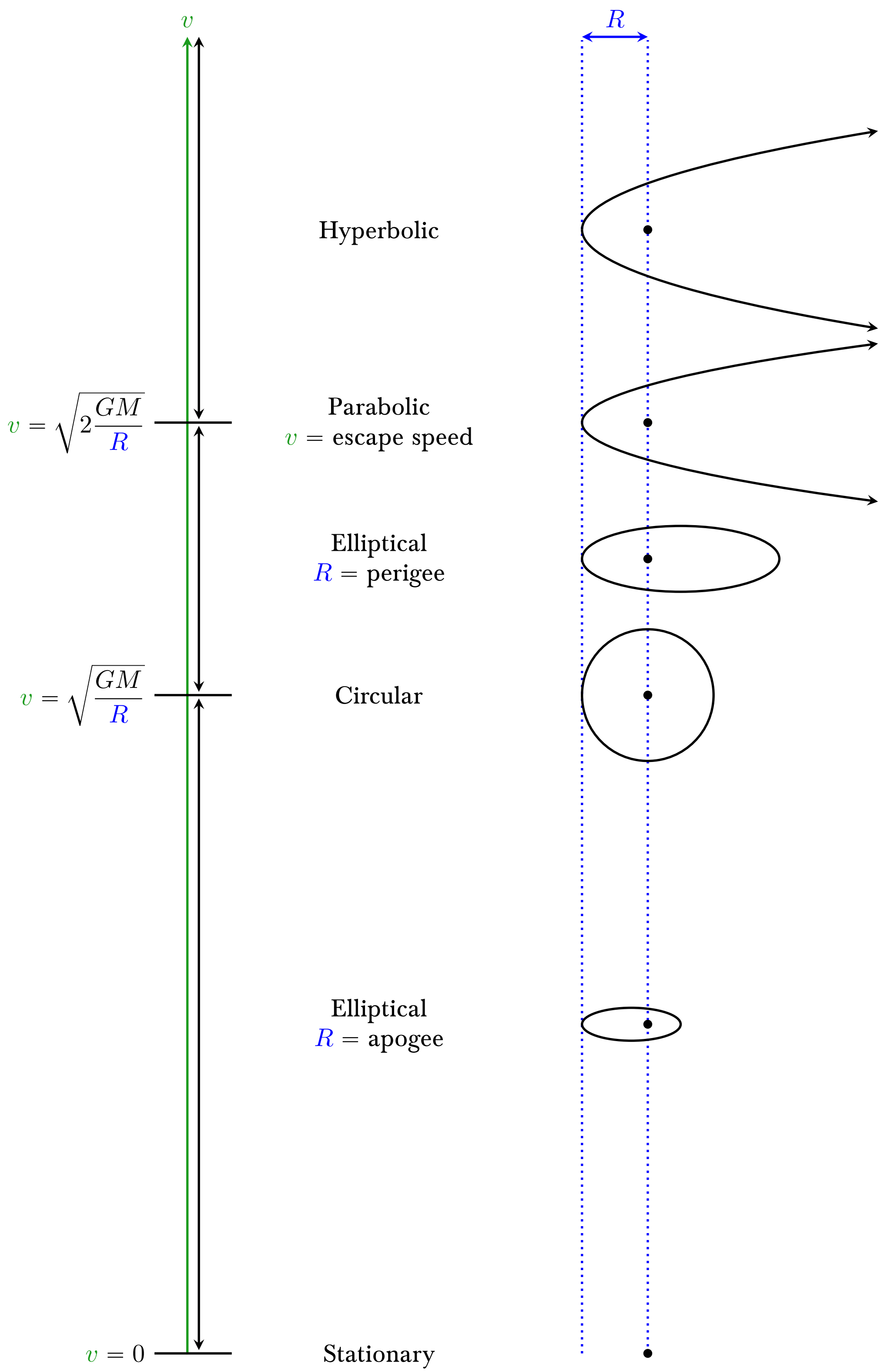
$$T^2 \propto R^3$$

$$\text{square of period} \propto \text{cube of radius}$$

The derivation shown there can be generalized to elliptical orbits without much difficulty, but is not required knowledge for the **ΣF⃗=ma⃗** Exam and will not be covered here.

7·4 ■ Non-Circular Orbits

The geometry of gravitational orbits depend on the ratio between v and $\sqrt{GM/R}$. Within the range of bound orbits, where $0 < v < \sqrt{2GM/R}$, the quantity R can represent either the apogee or the perigee, the furthest and closest distance between the orbiter and the orbited, respectively, or both.



Topic 8 ■ Dimensional Analysis

Every $\Sigma\vec{F}=m\vec{a}$ Exam features a few dimensional analysis questions, often masquerading as challenging questions from other topics. Recognizing these questions is crucial, as they can often be answered easily, or at least have several choices eliminated, before applying any other more time-consuming techniques.

A conspicuous clue that a question should be approached with dimensional analysis techniques is the answer choices having different dimensions. Some questions may have several choices with the same units. These can still make use of this dimensional analysis to eliminate answer choices, but still require other methods to narrow it down to a single answer.

Question. What is the period of a pendulum composed of a mass m hanging on a rope of length ℓ experiencing acceleration due to gravity g?

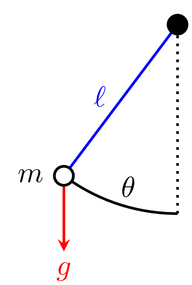
(A) $\sqrt{g/\ell}$

(B) $\sqrt{\ell/g}$

(C) $\sqrt{mg/\ell}$

(D) $\sqrt{\ell/mg}$

(E) $\sqrt{g\ell}$



Solution. All answer choices have unique units, so we will approach this question with dimensional analysis techniques. Since we want to find an expression for the period of the pendulum, we are looking for any answer choices with units of seconds.

There is only one mass in the question, and it can't cancel with itself, so $\sqrt{m}$ should not appear in our answer choice at all. Thus, we can eliminate choices (C) and (D).

We also need the lengths in g and ℓ to cancel each other out, so we can eliminate choice (E).

We can finish by manually checking the units of the remaining choices,

(A) $$\sqrt{\frac{g}{\ell}} \implies \left[\sqrt{\frac{g}{\ell}}\right] = \left(\frac{\mathrm{m}}{\mathrm{s}^2} \cdot \frac{1}{\mathrm{m}}\right)^{\frac{1}{2}} = \mathrm{s}^{-1}$$

(B) $$\sqrt{\frac{\ell}{g}} \implies \left[\sqrt{\frac{\ell}{g}}\right] = \left(\mathrm{m} \cdot \frac{\mathrm{s}^2}{\mathrm{m}}\right)^{\frac{1}{2}} = \mathrm{s}$$

Now, we can see the correct answer is (B).

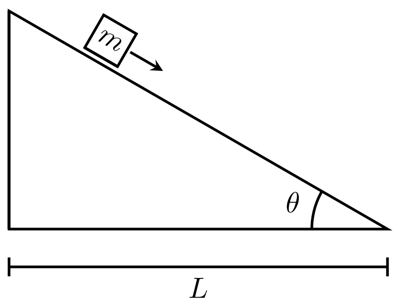
Topic 9 ■ Extremal Analysis

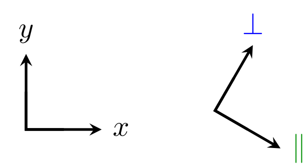
Many physics questions, both on and off the $\Sigma\vec{\text{F}}=\text{m}\vec{\text{a}}$ Exam, can benefit from the application of extremal analysis. Use of this technique is split into two distinct motifs.

9·1 ■ Vector Decompositions

A very frequent point of error and lag is the struggle of application of trigonometry to questions involving inclined planes, torques, and more. Perhaps the most common and illustrative lag point is recalling which components of a vector decomposition are the sine and cosine terms.

Question. Consider the inclined plane discussed in **Topic 2·2**. What is the normal force experienced by a mass placed upon this plane?

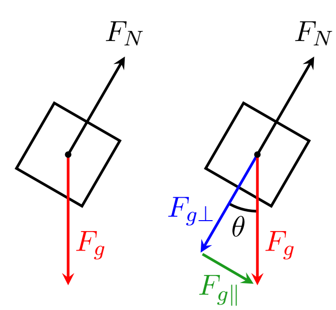




Solution. As before, let's make a free body diagram, and and decompose F_g into $\perp$ and $\parallel$ components.

Since the box is remaining along the surface of the plane, there is no acceleration in the $\perp$ direction. This requires

$$\vec{F}_{\text{net}\perp} = 0 = \vec{F}_N + \vec{F}_{g\perp} \implies F_N = -F_{g\perp}.$$



Now, all that remains to do is determine the magnitude of $\vec{F}_{g\perp}$ in terms of F_g. We know the magnitude is of the form

$$\vec{F}_{g\perp} = F_g \operatorname{trig}\theta.$$

Consider the extreme case where $\theta = 0$. In this situation, we can see that $\vec{F}_N$ and $\vec{F}_g$ are perfectly antiparallel, and thus equal in magnitude. Thus, in this extreme, $\vec{F}_{g\perp} = F_g$. So, whichever trigonometric function is required to find $\vec{F}_{g\perp}$ must have a value of $\operatorname{trig} 0 = 1$. A bonus of familiarity with the small angle approximation, discussed in **Topic 3·2**, is knowing

$$\sin\theta \approx \theta \implies \sin 0 = 0.$$

As such, the trig function required here must clearly be the cosine. Thus, we can confidently say

$$\vec{F}_{g\perp} = F_g \cos\theta \quad \text{and} \quad \vec{F}_{g\parallel} = F_g \sin\theta.$$

So,

$$F_N = F_g \cos\theta.$$

9.2 ■ Verifying Solutions

The second and more oft-used motif of extremal analysis is used for checking answers.

Question. Two masses, m_1 and m_2, are connected by a massless rope hung over a frictionless and massless pulley, in a configuration known as an Atwood's Machine. What is the downwards acceleration of m_1, if the local acceleration due to gravity is g?

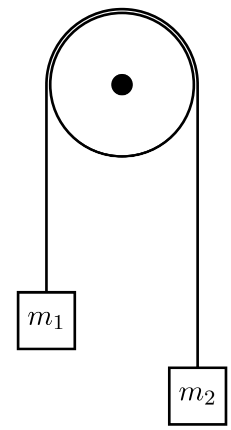
(A) $\left(\dfrac{m_1 - m_2}{m_1}\right) g$

(B) $\left(\dfrac{m_1 - m_2}{m_2}\right) g$

(C) $\left(\dfrac{m_1 - m_2}{m_1 + m_2}\right) g$

(D) $\left(\dfrac{m_2 - m_1}{m_1 + m_2}\right) g$

(E) $\left(\dfrac{m_1 + m_2}{m_1 - m_2}\right) g$



Solution. We can consider several extreme cases, and check which choices have the desired behaviors.

Answer Choice	Extreme 1 $m_1 = 0$ $a_1 = -g$	Extreme 2 $m_2 = 0$ $a_1 = +g$	Extreme 3 $m_1 = m_2 = m$ $a_1 = 0$
(A) $\left(\dfrac{m_1 - m_2}{m_1}\right) g$	$\left(\dfrac{0 - m_2}{0}\right) g = \text{undef.}$	$\left(\dfrac{m_1 - 0}{m_1}\right) g = +g$	$\left(\dfrac{m - m}{m}\right) g = 0$
(B) $\left(\dfrac{m_1 - m_2}{m_2}\right) g$	$\left(\dfrac{0 - m_2}{m_2}\right) g = -g$	$\left(\dfrac{m_1 - 0}{0}\right) g = \text{undef.}$	$\left(\dfrac{m - m}{m}\right) g = 0$
(C) $\left(\dfrac{m_1 - m_2}{m_1 + m_2}\right) g$	$\left(\dfrac{0 - m_2}{0 + m_2}\right) g = -g$	$\left(\dfrac{m_1 - 0}{m_1 + 0}\right) g = +g$	$\left(\dfrac{m - m}{m + m}\right) g = 0$
(D) $\left(\dfrac{m_2 - m_1}{m_1 + m_2}\right) g$	$\left(\dfrac{m_2 - 0}{0 + m_2}\right) g = +g$	$\left(\dfrac{0 - m_1}{m_1 + 0}\right) g = -g$	$\left(\dfrac{m - m}{m + m}\right) g = 0$
(E) $\left(\dfrac{m_1 + m_2}{m_1 - m_2}\right) g$	$\left(\dfrac{0 + m_2}{0 - m_2}\right) g = -g$	$\left(\dfrac{m_1 + 0}{m_1 - 0}\right) g = +g$	$\left(\dfrac{m + m}{m - m}\right) g = \text{undef.}$

We see that only (C) has the desired behavior for all three extremes, and must be the correct answer.

Topic 10 ■ Error Propagation

There are always one or two error propagation questions on each $\Sigma\vec{F}=m\vec{a}$ Exam. They are straightforward if you know the formulae and how to apply them.

If measured quantities x and y have Gaussian uncertainties Δx and Δy, which is always true, but rarely explicitly stated on the $\Sigma\vec{F}=m\vec{a}$ Exam, uncertainties of multiple measurements follow the rules presented in this Topic.

Note that $(\Delta x)^2 \neq \Delta(x^2)$. Don't write Δx^2 without parentheses under any circumstances.

10·1 ■ Basic Error Propagation

Additive Uncertainties. If you have two uncertain quantities which are added,

$$\Delta(x+y) = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

Multiplicative Uncertainties. The product of two uncertain quantities has uncertainty

$$\Delta(xy) = \sqrt{(x\Delta y)^2 + (y\Delta x)^2}.$$

Uncertainty of Exponentiated Quantities. The uncertainty of an uncertain quantity raised to a power is given by

$$\Delta(x^a) = |a|x^{a-1}\Delta x.$$

10·2 ■ Multiple Trials and Weighted Results

Error propagation for multiple trials and weighted results works the same way as additive uncertainties, except the final uncertainty is divided by the square root of the number of trials or total weighting.

Question. Two experimenters, Xavier and Yvonne, performed the same experiment and obtained results $x \pm \Delta x$ and $y \pm \Delta y$, respectively. Notably, Yvonne obtained her results over two trials, so we wish to weight her measurement twice as much. What is the uncertainty of their collective result?

Solution. In this case, we would say

$$\Delta(x+2y) = \frac{1}{\sqrt{3}}\sqrt{(\Delta x)^2 + 2(\Delta y)^2}.$$

10·3 ■ Relative Uncertainties

Another favorite method for propagating uncertainties is the use of relative uncertainties. When two uncertain quantities are multiplied or divided, their relative uncertainties add in quadrature. For example, if you measure a displacement $x \pm \Delta x$ taking place over a time $t \pm \Delta t$, the velocity should be reported as $v = x/t$, with uncertainty Δv as given by

$$\frac{\Delta v}{v} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta t}{t}\right)^2}.$$

10·4 ■ Chaining Rules

Question. Consider an experiment to measure the acceleration due to gravity g, accomplished by measuring the length and period of a simple pendulum. We measured $L \pm \Delta L$ and $T \pm \Delta T$, respectively. What is $g \pm \Delta g$?

Solution. We know that $g = 4\pi^2 L/T^2$. So, our first step is to find $\Delta(T^{-2})$. We use the rule for uncertainty of exponentiated quantities to say

$$\Delta\left(T^{-2}\right) = |-2| T^{(-2-1)} \Delta T = 2T^{-3}\Delta T.$$

Next, we want to find $\Delta(LT^{-2})$, which is

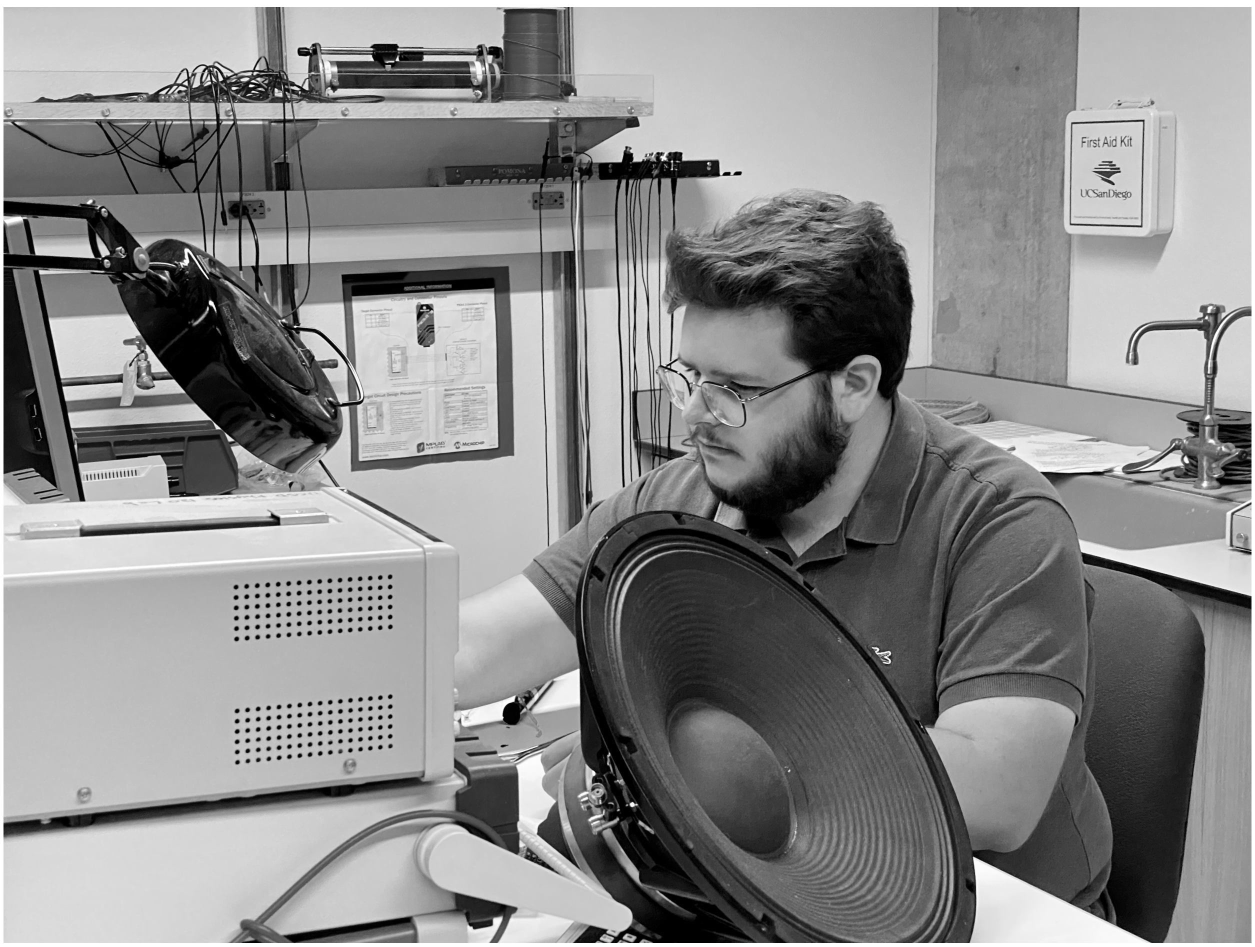
$$\Delta(LT^{-2}) = \sqrt{\left[L\Delta\left(T^{-2}\right)\right]^2 + \left[\left(T^{-2}\right)\Delta L\right]^2} = \sqrt{\left[2LT^{-3}\Delta T\right]^2 + \left[\left(T^{-2}\right)\Delta L\right]^2}$$

from the rule for multiplicative uncertainties. Lastly, we tack on our scalar factor, and see

$$\Delta g = 4\pi^2 \sqrt{\left[2LT^{-3}\Delta T\right]^2 + \left[\left(T^{-2}\right)\Delta L\right]^2}.$$

About the Author

This guide was written by Jacob Saret, director of Saret & Co. Education. He began developing this material when teaching his acclaimed $\Sigma\vec{F}=m\vec{a}$ Physics course at the San Diego Math Circle for the 2019 $\Sigma\vec{F}=m\vec{a}$ Exam season. Since then, he has developed and expanded his original program and delivered it year after year to the students of the San Diego Math Circle. Now, it is available in its most refined form to anyone interested in preparing for the $\Sigma\vec{F}=m\vec{a}$ Exam.



The author in his natural habitat, the Physics 120 laboratory in Mayer Hall at UCSD.

Visit `saret.co/about` to learn more about the author's personal background.