Rotational Dynamics

Two key facets of rotation-based problems.

1. Rotational dynamics are strongly analagous to linear dynamics.

2. Rotational and linear motion often need to be treated simultaneously.

First, analogies.

$F \to N$: $N = r \times F = rF$ if $r \perp F$.

$a \to \alpha$: $\alpha = a/r$

$v \to \omega$: $\omega = v/r$

$x \to \theta$: $\theta = x/r$

$m \to I$: $I = mr^2$

$K \to K$: $K = \frac{1}{2} I \omega^2$

These allow us to convert basic kinematics to rotation, e.g.

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 \Rightarrow \theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2.$$

Example.

A ball (m, R) is gently dropped on a surface, while spinning with angular velocity ω_0. Once it is rolling without slipping on the surface, how fast is it moving? No energy is lost to the surface. $(I_{sphere} = 2/5\, mR^2)$.

Solution.

Let's take the energy approach.

$E_0 = \frac{1}{2}\left(\frac{2}{5} mR^2\right)\omega_0^2.$

When it's rolling without slipping, $v = \omega R$. So,

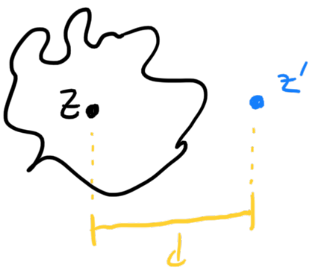
$E_f = \frac{1}{2} mv^2 + \frac{1}{2}\left(\frac{2}{5} mR^2\right)\omega^2 = \frac{7}{10} mv^2.$

Equating E_0 and E_f,

$\frac{1}{5} mR^2\omega_0^2 = \frac{7}{10} mv^2.$ So, $\boxed{v = \sqrt{2/7}\, R\omega_0}.$

How would this change if it was a hollow sphere? $(I_{h.s.} = \frac{2}{3} mR^2)$

Parallel axis theorem.

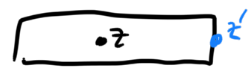


$$I_z' = I_z + m d^2$$

for any two parallel axes.

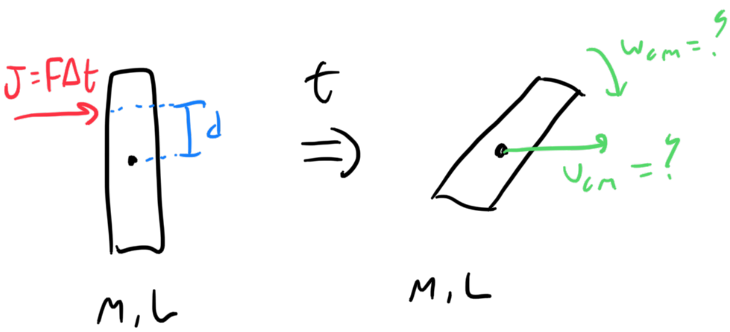
Example.

Rod about center vs. edge.



$$I_z' = \frac{1}{12} mL^2 + m\left(\frac{L}{2}\right)^2 = \frac{1}{3} mL^2.$$

Example from MIT.



Linear component:

$$J = F\Delta t = Ma\Delta t = M v_{cm}$$

so $v_{cm} = J/M$

independent of d!

Angular component:

$$J_N = F\Delta t\, d = I\alpha\Delta t$$

$$= I\omega_{cm} = \frac{1}{12} ML^2 \omega_{cm}$$

$$\Rightarrow \quad \omega_{cm} = 12 Jd/ML^2$$

Classmate question.

"Does d affect the energy imparted on the rod?"

Let's find out.

$$\Delta E = F \Delta y = J \, \Delta y / \Delta t$$

If Δt is small, $\Delta y \approx d \Delta \theta + \Delta x$.

So, $\alpha = Jd / I \Delta t$ and $a = J / M \Delta t$,

then $\Delta \theta = \frac{1}{2} \alpha \Delta t^2 = \dfrac{J d \Delta t}{2I}$ and $\Delta x = \frac{1}{2} a \Delta t^2 = \dfrac{J \Delta t}{2M}$

$$\Rightarrow \Delta y = \frac{J \Delta t}{2M}\left(1 + \frac{d^2}{L^2}\right)$$

So $\boxed{\Delta E = \dfrac{J^2}{2M}\left(1 + \dfrac{d^2}{L^2}\right)}$ which, sensibly, depends on d.