Outline:

- Black hole Astrophysics.
- Dark stars – A Newtonian Derivation.
- General Relativity – Crash course.
- Time Dilation.
- Gravitational Lensing.

• Black Holes in Astrophysics

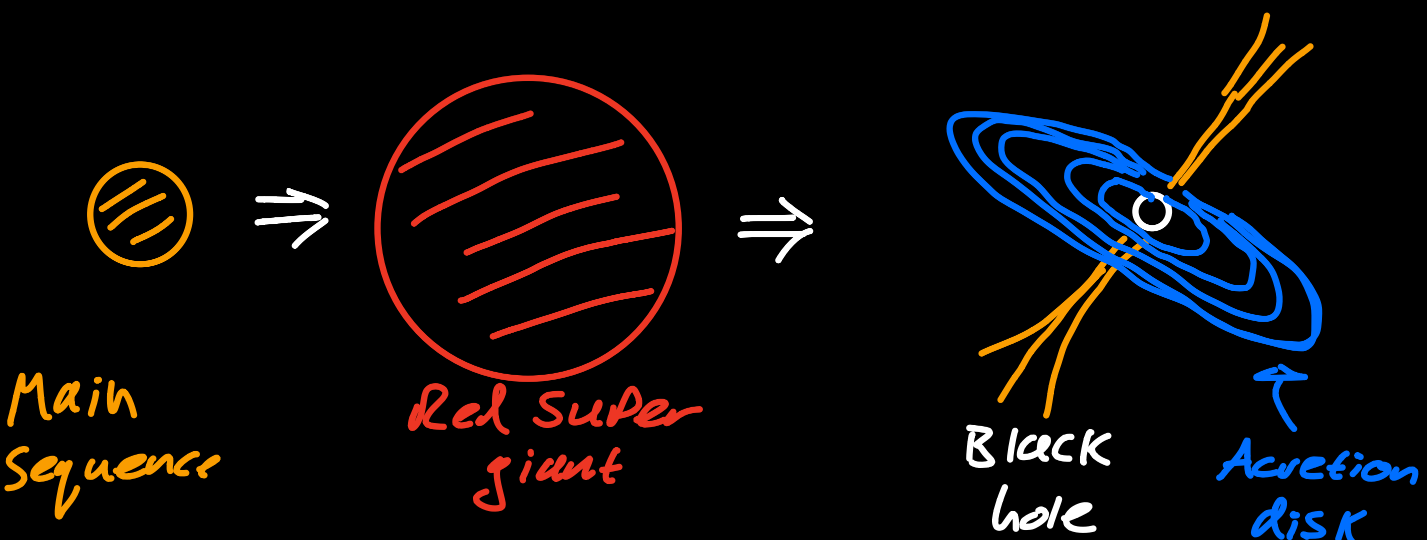
• Different Mass Black Holes :

– Stellar Mass Black Holes :

$$3M_\odot \lesssim M \lesssim 30M_\odot$$

End state of stars with core more massive $\sim 2.17 M_\odot$ (in theory)

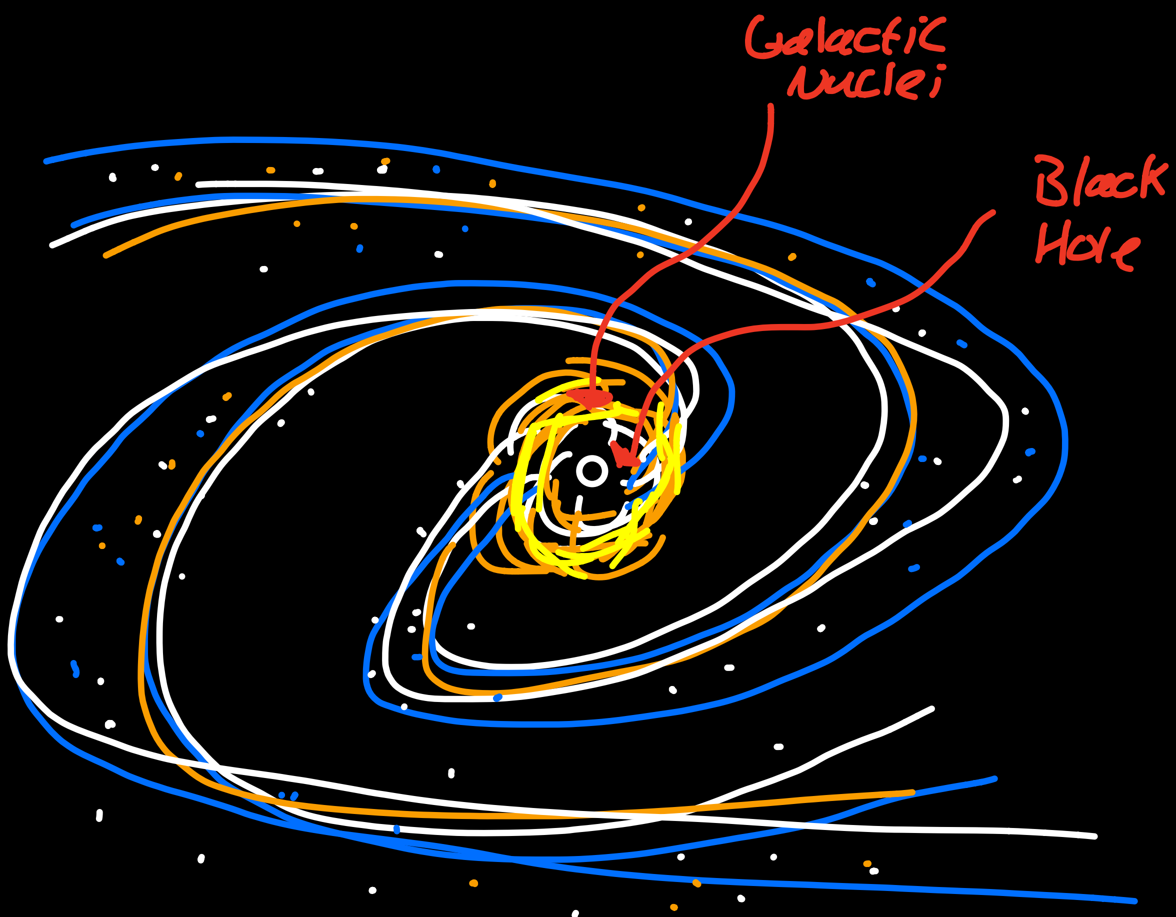
($2.17\ M_\odot \rightarrow$ Tolman-Oppenheimer-Volkoff limit)



- Supermassive Black Holes:

$$10^5 M_\odot \lesssim M \lesssim 10^9 M_\odot$$

All moderately sized galaxies are thought to have a supermassive BH at the centre. These BHs are thought to be crucial to galaxy formation.



The Milky Way → BH = Sagittarius A^*

$$M \sim 2.6 \times 10^6 M_\odot$$

- Primordial Black Holes:

$$M \lesssim M_\odot$$

BH's formed (in theory) in the very early universe, when matter densities were high.

In practice, the PBH's surviving today would have to be more massive than $\sim 10^{11}$ kg

($\sim 10^{-19} M_\odot$) else they would have decayed by present day.

PBH's could contribute to DM in universe.

Currently no evidence for PBH'S.

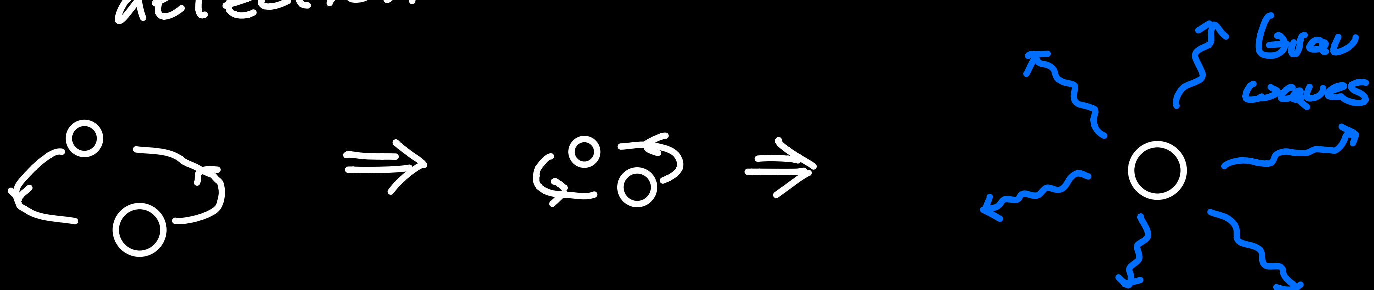
• Observing Black Holes:

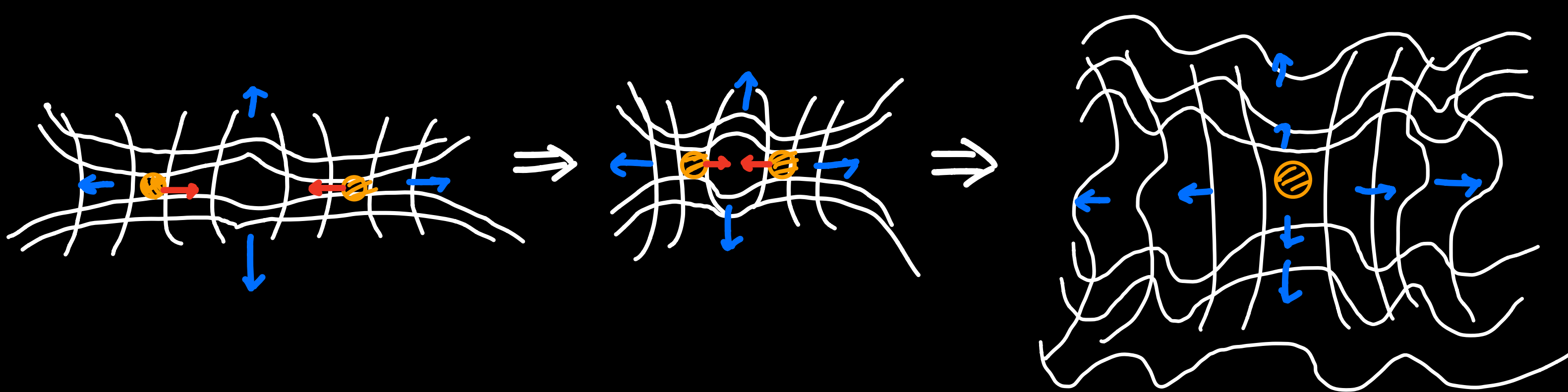
'Well, the thing about a black hole - its main distinguishing feature - is it's black. And the thing about space, your basic space colour, is black'

- Holly, Red Dwarf S3 EP2

- Gravitational Radiation:

LIGO first detection $\longrightarrow$ 14^{th} Sept 2015





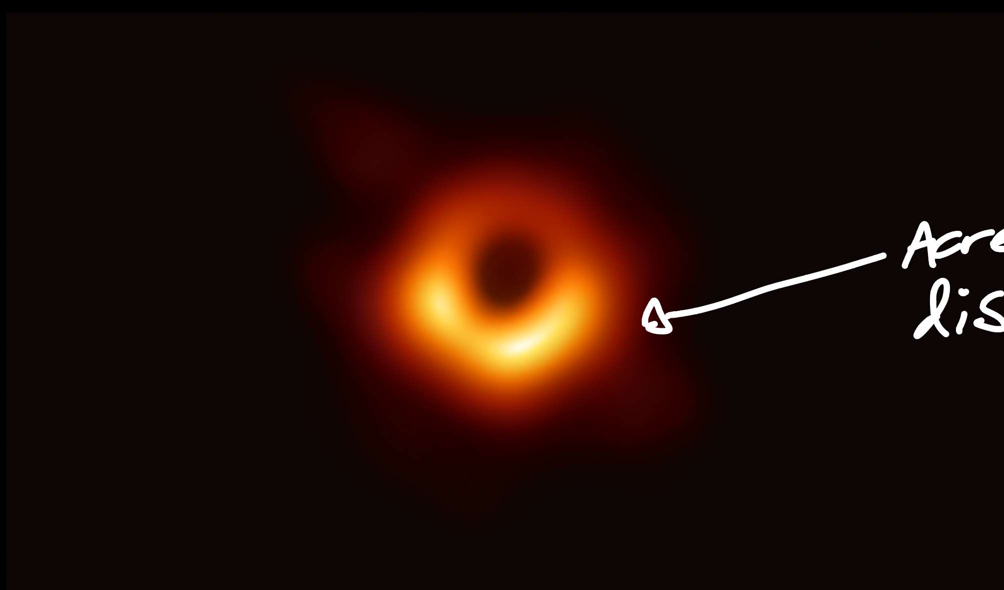
first detection Event:

Two black holes with masses $\sim 35 M_\odot$ and $\sim 30 M_\odot$ merging.

Releasing energy $\sim 3 M_\odot \times c^2$

- Direct Imaging:

Image Published $\longrightarrow$ 10th April 2019

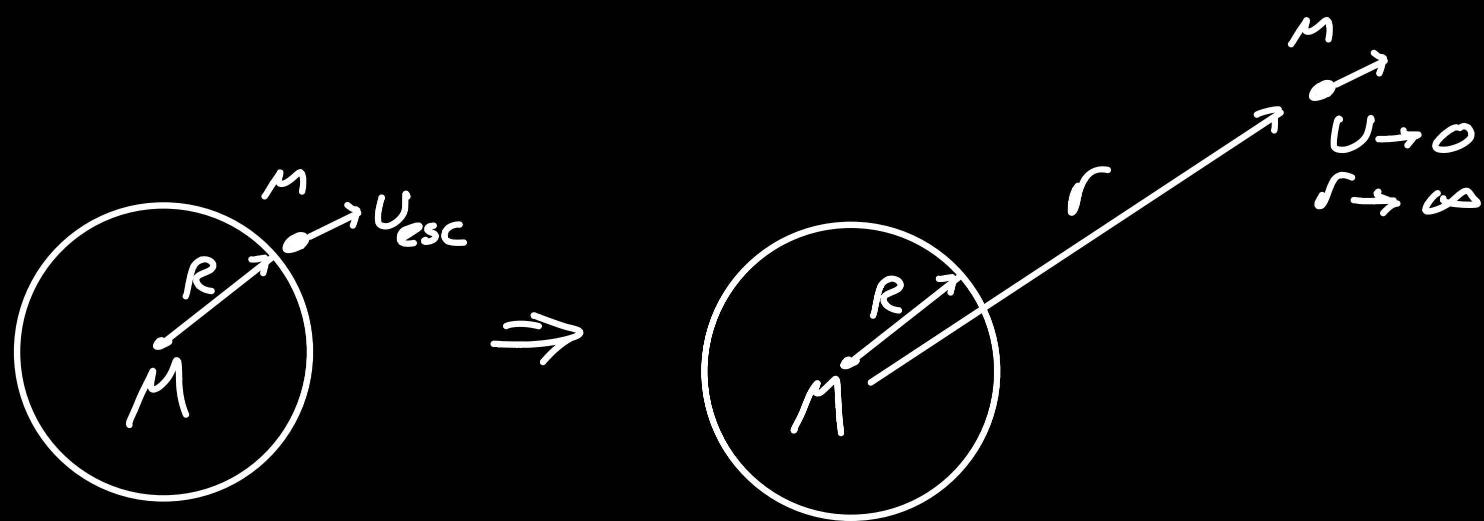


Acretion disk

- Dark Stars:

circa ~ 1783

ordinary mass with escap velocity > c



$$-\frac{GMm}{R} + \frac{1}{2} m U_{esc}^2 = -\frac{GMm}{r}\Big]_{r\to\infty} + \frac{1}{2} m U^2\Big]_{U\to 0}$$

$$\Rightarrow U_{esc}^2 = \frac{2GM}{R}$$

for a dark star, $U_{esc} = c$

$$\Rightarrow R = \frac{2GM}{c^2}$$ ← 'Gravitational radius'

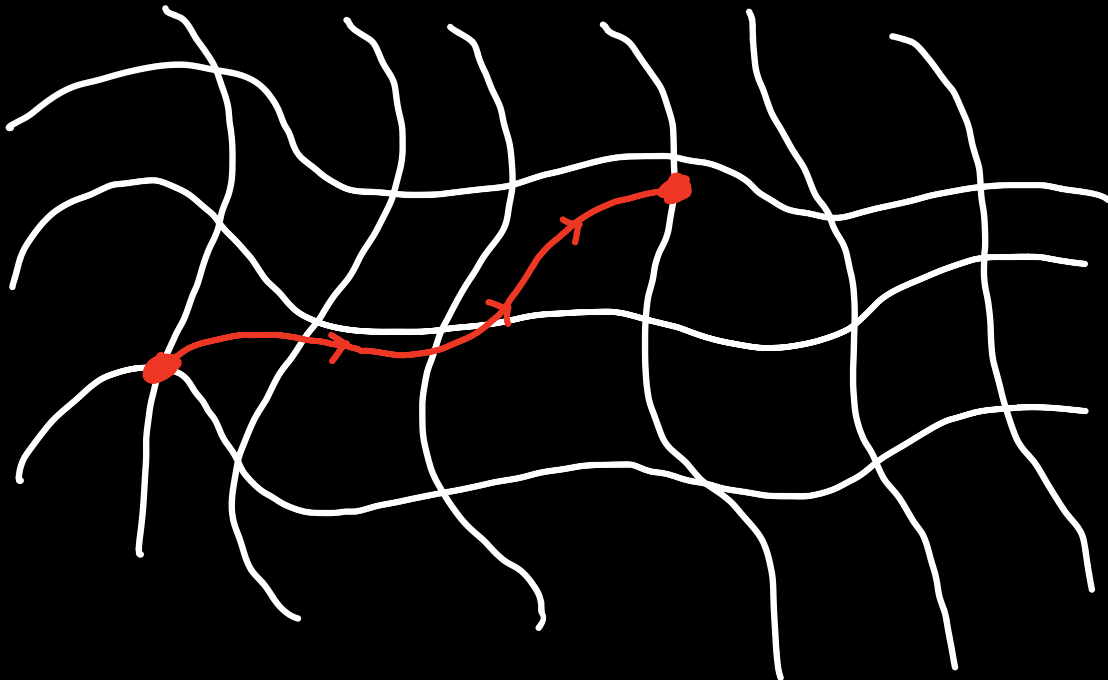
Actually same as schwarzschild radius in GR

- General Relativity - Crash Course:

Equivalence Principle $\longrightarrow$ free falling observers feel no effects of gravity

Allows us to equate effects of gravity with 'free falling' paths through spacetime.

We call these paths $\rightarrow$ Geodesics



Geodesic path will depend on curvature of the space time.

$$\boxed{\text{Gravity} \simeq \text{Curvature}}$$

$$\boxed{\text{Matter/Energy} \longleftrightarrow \text{Gravity}}$$

$$\boxed{\begin{array}{l}\text{Curvature} \\ \text{of Space time}\end{array} \sim \text{Matter/Energy}}$$

$T^{\mu\nu}$ → Describes Matter/Energy distribution sourcing gravitation

Rank-2 tensor

Multiple ways to describe curvature.

To equate to $T_{\mu\nu}$, we need a rank-2 tensor describing curvature.

October 1915 Einstein tried

$$R^{\mu\nu} \sim T^{\mu\nu}$$

Ricci curvature tensor

<u>But this is wrong</u>

why? — from SR.

$$\underbrace{\partial_\mu T^{\mu\nu} = 0}$$

contains statements of conservation of Energy and momentum

$\Rightarrow$ In GR:

$$\nabla_\mu T^{\mu\nu} = 0$$

Covariant derivative

But $\nabla_\mu R^{\mu\nu} \neq 0$ in general

we need a rank-2 description of curvature which has ∇_μ (Tensor) $= 0$

A crucial Identity from differential geometry:

Ricci scalar

$R = g^{\mu\nu} R_{\mu\nu}$

$$\nabla_\mu \underbrace{\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right)}_{\text{Rank - 2}} \equiv 0$$

Bianchi identity (1889)

November 1915 Einstein wrote down

$$G_{\mu\nu} \sim T_{\mu\nu}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

Can show by comparison with Newtonian theory:

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

Einstein's field equations

$R_{\mu\nu} \sim$ non linear combinations of second derivatives of $g_{\mu\nu}$ $\longrightarrow$ solve equations $\Rightarrow$ find metric $g_{\mu\nu}$

Equations are a system of non linear 2^{nd} order PDE's (very hard to solve)

December 1915

Schwarzschild found the first analytic solution $\longrightarrow$ Schwarzschild Metric

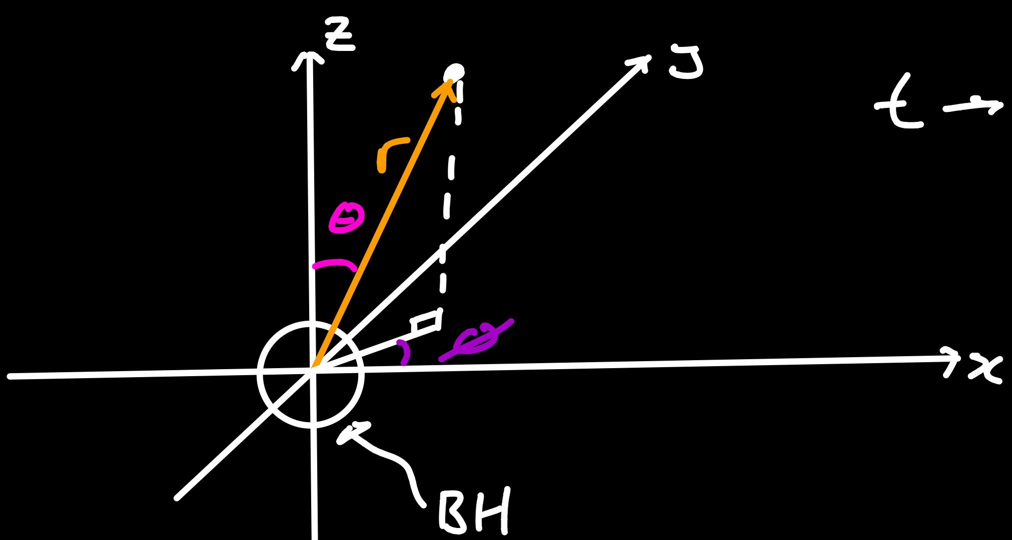
Describes spacetime outside of a spherically symmetric matter distribution (stars, planets, BH's)

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)dr^2 + \ldots$$

$$\ldots + r^2 d\Omega^2 = \underbrace{-c^2 d\tau^2}_{\text{Proper time of observer at } (r,\theta,\phi)}$$

$\uparrow$ Invariant space time interval

where $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$



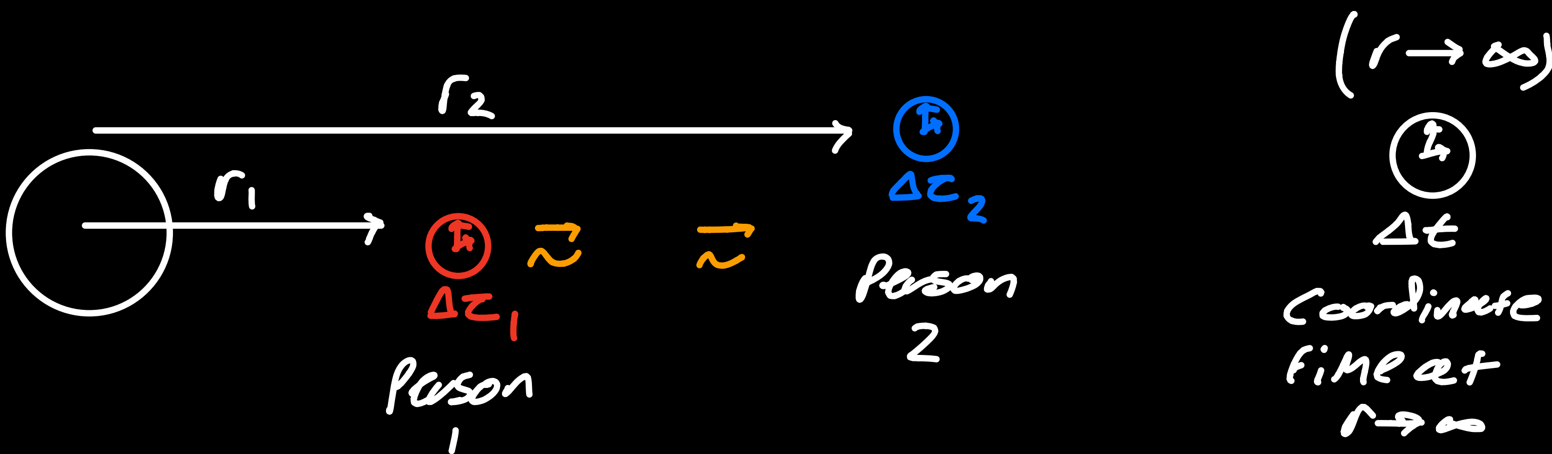
$t \rightarrow$ time coord of observer very far from BH

• Time Dilation:

Replace $\frac{2GM}{c^2} = R_s$, schwarzschild radius
(Radius of event horizon)

$$-c^2 d\tau^2 = -\left(1-\frac{R_s}{r}\right)c^2 dt^2 + \left(1-\frac{R_s}{r}\right)dr^2 + \dots$$

$$\dots + r^2 d\Omega^2$$



$$-c^2 \Delta\tau_1^2 = -\left(1-\frac{R_s}{r_1}\right)\Delta t^2$$

$$-c^2 \Delta\tau_1^2 = -\left(1-\frac{R_s}{R_1}\right)\Delta t^2$$

$$\Rightarrow \quad \Delta\tau_1 = \sqrt{\frac{1-\frac{R_s}{r_1}}{1-\frac{R_s}{r_2}}}\;\Delta\tau_2$$

$$\Delta\tau_1 < \Delta\tau_2$$

Black Holes warp time.

The closer you are to a black hole, the less time you experience.

(your clock ticks slower)

what happens as person 1 approaches the horizon?

$$r_1 \to R_s$$

$$\Rightarrow \quad \Delta\tau_1 \to 0$$

while $\Delta\tau_2$ is finite

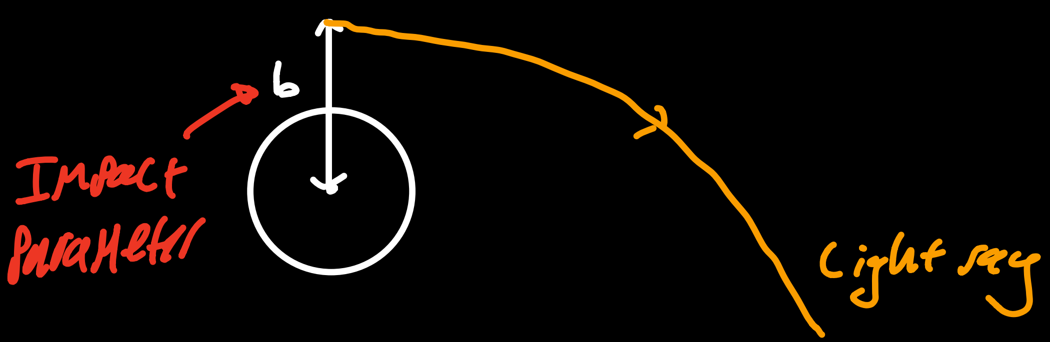
What would Person 1 See?

They would see an event that would take a finite time measured by person 2 occur in a $\to 0$ ammount of time.

If you are near a black hole. time passes much faster further away.

• Gravitational Lensing:

Light rays passing near BH's (or indeed other massive objects) are deflected.



$$\mathcal{L} = \underset{\text{for light}}{0} = -\left(1-\frac{R_s}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{R_s}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \dots$$

$$\dots + r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

• $\frac{d}{d\lambda}\left(2r^2\dot{\phi}\right) = 0$

$$\Rightarrow \dot{\phi} = \frac{J}{r^2}$$

- $$\frac{d}{d\lambda}\left[-2c^2\left(1-\frac{R_s}{r}\right)\dot{t}\right]=0$$

$$\Rightarrow \quad \dot{t}=-\frac{E}{c^2}\left(1-\frac{R_s}{r}\right)^{-1}$$

$\Rightarrow$

$$0=-c^2\left[\frac{E}{c^2}\left(1-\frac{R_s}{r}\right)^{-1}\right]^2\left(1-\frac{R_s}{r}\right)+\dots$$

$$\dots+\dot{r}^2\left(1-\frac{R_s}{r}\right)^{-1}+\frac{J^2}{r^2}$$

$$\dot{r}^2+\frac{J^2}{r^2}\left(1-\frac{R_s}{r}\right)=\left(\frac{E}{c}\right)^2$$

$$\boxed{\dot{r}^2+\frac{J^2}{r^2}-\frac{R_sJ^2}{r^3}=\left(\frac{E}{c}\right)^2}$$

Times by $\times\frac{1}{2}$ and compare with equation for orbits we found while studying mercury: using $R_s = \frac{2GM}{c^2}$

for Light Deflection

$$\frac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2} - \frac{GMJ^2}{c^2r^3} = \frac{1}{2}c^2\left(\frac{E}{c^2}\right)^2$$

for Matter orbiting

$$\frac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2} - \frac{GM}{r} - \frac{GMJ^2}{c^2r^3} = \frac{1}{2}c^2\left[\left(\frac{E}{c^2}\right)^2 - 1\right]$$

Newtonian potential term not present for light

Light deflection/gravitational lensing is a purely GR effect.

$$\underbrace{\frac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2} - \frac{GMJ^2}{c^2 r^3} = \frac{1}{2}c^2\left(\frac{E}{c^2}\right)^2}$$

we do to this as we did to the matter orbit equation last time.

Change variables $u = \frac{1}{r}$, and then differentiate both sides w.r.t ϕ:

$$\Rightarrow \boxed{\frac{d^2u}{d\phi^2} + u - \frac{3}{2}R_s u^2 = 0}$$

In contrast to result we derived for Mercury:

$$\frac{d^2u}{d\phi^2} + u - \frac{3GM}{c^2}u^2 - \frac{GM}{J^2} = 0$$

This is missing

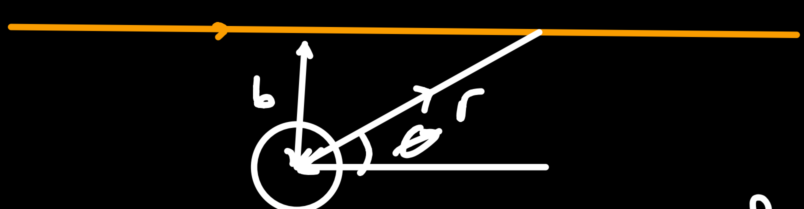
$$\frac{d^2u}{d\phi^2} + u - \frac{3}{2}R_s u^2 = 0$$

R_s will be a small length scale for the problem

Treating $R_s \ll b$: $\Rightarrow b = \frac{B}{\varepsilon}$ for some small $\varepsilon > 0$

If we ignore this $\frac{3}{2}R_s u^2$ GR term;

$$\frac{d^2u}{d\phi^2} + u = 0 \Rightarrow u = \frac{1}{b} \text{ at } \phi = \frac{\pi}{2}$$



$\Rightarrow u = \frac{1}{b}\sin\phi$

no deflection

Now include GR term as a small perturbation the equation:

$$\frac{d^2u}{d\phi^2} + u - \frac{3}{2}R_s u^2 = 0$$

$$u = \frac{1}{b}\sin\phi + \varepsilon^2 f(\phi)$$

$$= \frac{\varepsilon}{B}\sin\phi + \varepsilon^2 f(\phi)$$

ε^1 order : $u = \frac{1}{b}\sin\phi$

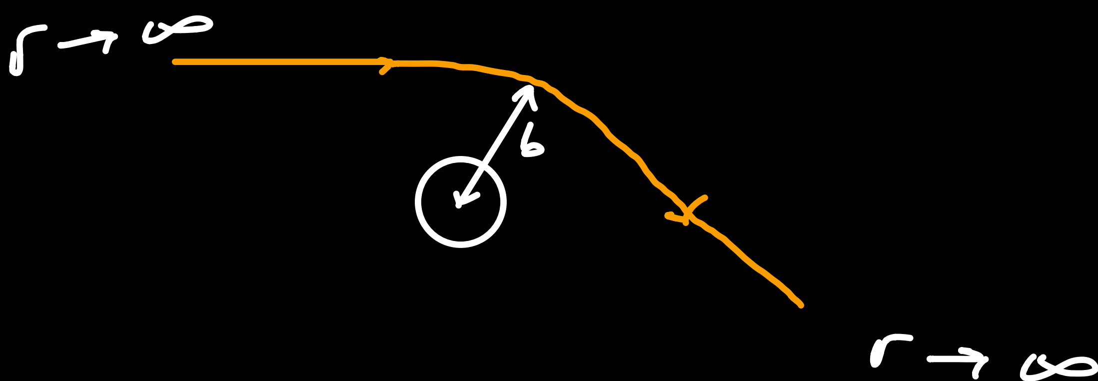
ε^2 order : $f''(\phi) + f(\phi) - \frac{3}{2}R_s\left(\frac{1}{B}\sin\phi\right)^2 = 0$

$$\Rightarrow f''(\phi) + f(\phi) - \frac{3}{2}R_s\frac{1}{B^2}\sin^2\phi = 0$$

$$\Rightarrow f''(\phi) + f(\phi) - \frac{3}{4}R_s\frac{1}{B^2}(1-\cos 2\phi) = 0$$

$$f''(\phi) = \frac{3}{4} R_s \frac{1}{B^2} + \frac{1}{4} R_s \frac{1}{B^2} \cos 2\phi$$

$$\Rightarrow u = \frac{1}{b} \sin\phi + \frac{1}{4} R_s \frac{1}{b^2} \left(3 + \cos 2\phi \right)$$



$r \to \infty$

$\Rightarrow u \longrightarrow 0$

for small deflections

$$\sin\phi \sim \phi$$
$$\cos\phi \sim 1$$

$$\Rightarrow 0 = \frac{1}{b} \Delta\phi + \frac{1}{4} R_s \frac{1}{b^2} (3+1)$$

$$|\Delta\phi| = \frac{R_s}{b}$$

factor of 2 off....

will find another time