Black Holes & Entropy

useful reading:

'Concepts in thermal Physics' - Blundell & Blundell

'Black Holes and the second law' - Bekenstein (1972)

Some classical Thermodynamics:

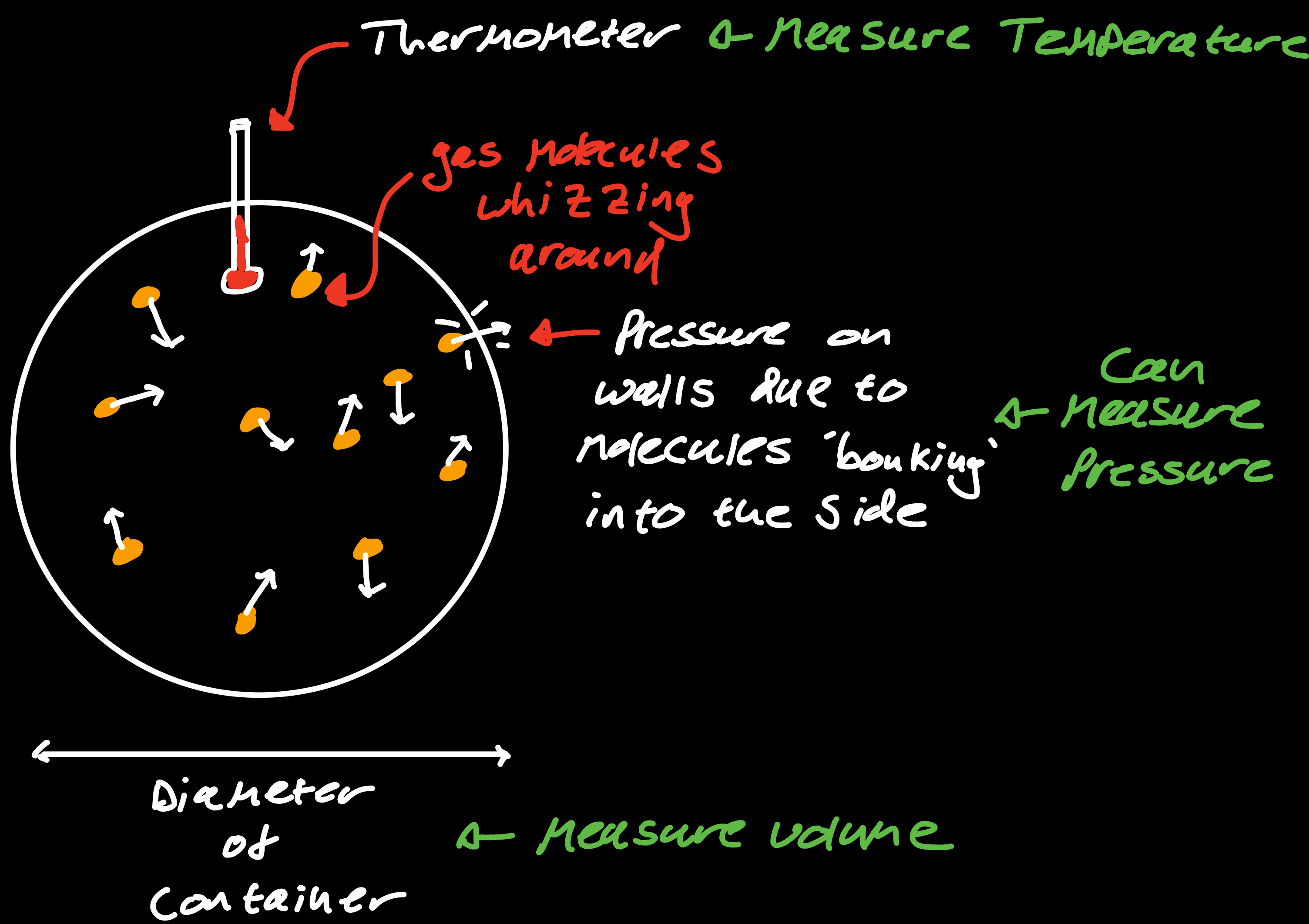
- Macrostates vs Microstates:

Macrostate → A description of the system involving macroscopic measurable quantities, such as Temperature, Volume, pressure, Charge, Angular Momentum etc...

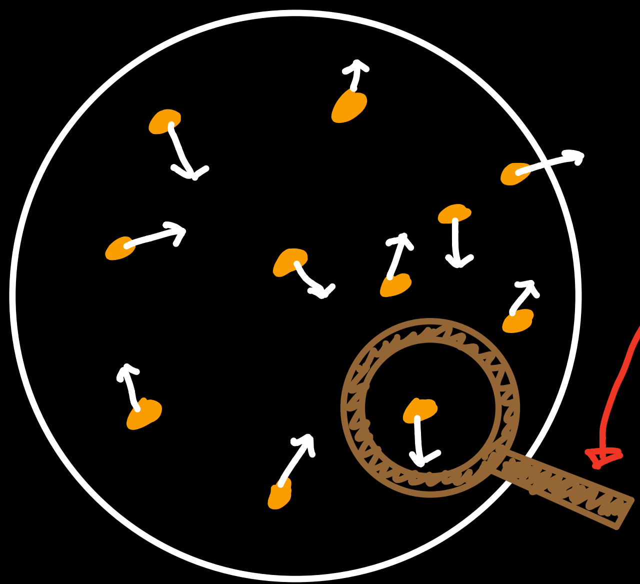
Microstate → The microscopic configurations of atoms/particles/fields that give rise to a particular Macrostate.

E.g. for a gas of particles in a sealed isolated container: we can measure the volume of the container, the pressure on the container walls, and the temperature.

Macrostates:



Microstates:



'Zoom in' on each particle:
Microstates of this classical system correspond to the complete labelling of every particles position (x, y, z) and velocity (v_x, v_y, v_z)

It's clear from this example, that there are many possible microstates for a given macrostate.

Through kinetic theory, we understand temperature roughly as the average kinetic energy per particle in the gas

→ There are lots of different ways we can distribute the particles individual speeds to give the same temperature.

- Entropy:

Entropy measures the number of microstates that correspond to a particular macrostate.

In this sense it is a measure of the ignorance we have about the microphysics that leads to macroscopic properties.

For the simplest system with fixed total energy (the microcanonical ensemble) the entropy is given by

$$S = \ln \underbrace{\Omega}_{\text{number of microstates in the given macrostate}}$$

for more complicated systems (which are allowed to exchange heat and particles with their surroundings)

$$S = \sum_{i=\text{macro-states}} \ln \Omega_i = -\sum_{i=\text{macro-states}} P_i \ln P_i$$

Probability system will be in certain macrostate

- The first & second laws of thermodynamics:

1st law:

$$\underbrace{dU}_{\text{change in internal energy}} = \underbrace{SdT}_{\text{Energy gained through heat}} - \underbrace{PdV}_{\text{work done by system on surroundings}}$$

2nd law: Total Entropy of an isolated system can only increase

$$dS \geq 0$$

- Black Hole Macrostates:

Classically, black holes are described by three numbers:

the BH's mass → M
the BH's charge → Q
the BH's angular momentum → J

A Black hole can only exist when:

$$M^2 - \left(\frac{J}{M}\right)^2 - Q^2 \geq 0$$

The 'no hair theorem' forbids the description of a black hole by any other independant quantities.

Note: This theorem is a classical GR result, we don't expect this to hold in a full theory of quantum gravity.

• Black Hole Microstates (?):

String theory → offers a way to count microstates of Supersymmetric BH's in 5D

A very interesting topic

• Black Hole Entropy:

Appears to be a problem with the 2nd law, and BH's.

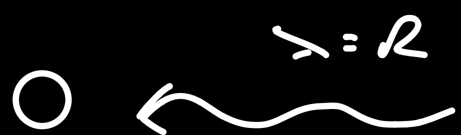
Imagine dropping some source of entropy into a black hole. This decreases the entropy of the universe outside.

However, as a BH is described by only three numbers, we cannot go about calculating it's entropy.

For this to be resolved, a Black hole must carry some entropy. But this means it must also have a temperature.

We will argue a value for the entropy of a Black Hole.

• A Classical (cheaty) Derivation:



$$\delta E = \frac{\hbar c}{R}$$

$$\delta M = \frac{\hbar}{Rc}$$

$$\Rightarrow \delta R = \frac{2G}{c^3}\frac{\hbar}{R}$$

$$\delta A = \frac{4Gh}{c^3} \Rightarrow A = \frac{4Gh}{c^3} N$$

$$N = \frac{Ac^3}{4Gh}$$ number of 'bits'

$$S = \frac{Ac^3}{4hG}$$

- A semiclassical (slightly less cheaty) derivation.

Unruh Effect: Accelerating observers measure a thermal 'bath' of particles, not seen by non-accelerating observers.

Let's try and convince ourselves of this through a simplified argument.

$$a^\mu(\tau) = \frac{d^2 x^\mu}{d\tau^2}$$

$$x^\mu = (ct, \underline{x})$$

$$\frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\left(c, \underline{u}(\tau)\right) = \gamma(c, \underline{u})$$

$$a^\mu(\tau) = \frac{d^2 x^\mu}{d\tau^2} = \frac{d\gamma}{d\tau}\left(c, \underline{u}(\tau)\right) + \gamma^2\left(0, \underline{a}(\tau)\right)$$

$$|a(\tau)|^2 = a^\mu a_\mu = |a_0|^2$$

3-Acceleration measured in instantaneous comoving reference frame of Particle

$$a^\mu(\tau) = \left(c\frac{d^2 t}{d\tau^2}, \frac{d^2 x}{d\tau^2}, 0, 0 \right)$$

$$a^\mu a_\mu = -c^2\left[\frac{d^2 t}{d\tau^2}\right]^2 + \left[\frac{d^2 x}{d\tau^2}\right]^2 = a_0^2$$

Solutions :

$$\left.\begin{aligned} t(\tau) &= \frac{c}{a_0}\sinh\left(\frac{a_0\tau}{c}\right) \\ x(\tau) &= \frac{c^2}{a_0}\cosh\left(\frac{a_0\tau}{c}\right) \end{aligned}\right\}$$ world line of uniformly accelerating particle

$$\underbrace{U(\tau) = \frac{dx}{dt}}_{\text{velocity seen in non accelerating frame}} = c\tanh\left(\frac{a_0\tau}{c}\right)$$

$$\begin{pmatrix} \bar{\omega} \\ \bar{k} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} \omega/c \\ k \end{pmatrix} \qquad k = \frac{\omega}{c}$$

$$\frac{\bar{\omega}}{c} = \gamma\left(\frac{\omega}{c} - \beta k\right)$$

$$\bar{k} = \gamma\left(k - \beta \frac{\omega}{c}\right)$$

where $\beta = \frac{v}{c} = \tan\left(\frac{a_0 \tau}{c}\right)$

$$\Rightarrow \quad \bar{\omega}(\tau) = \omega e^{-\frac{a_0 \tau}{c}} \qquad \text{for } +x$$

$$\bar{\omega}(\tau) = \omega e^{\frac{a_0 \tau}{c}} \qquad \text{for } -x$$

Waves with time dependent phase

Take fourier transform (extract frequency spectrum)

$$\phi(t) = \int_0^{\tau} \omega(\tau)\, d\tau = \frac{\omega c}{a_0} e^{\frac{a_0 \tau}{c}}$$

$$I = \int_{-\infty}^{\infty} d\tau \; e^{i\Omega t} \; e^{i\left(\frac{\omega c}{a_0} e^{\frac{a_0 \tau}{c}}\right)}$$

$$u = e^{\frac{a_0 \tau}{c}}$$

$$I = \frac{c}{a} \int_0^{\infty} du \; u^{\left(i\Omega \frac{c}{a} - 1\right)} e^{i\left(\frac{\omega c}{a_0}\right) u}$$

Recall definition of gamma function:

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x}\, dx$$

$$\Rightarrow \quad I = \frac{c}{a} \Gamma\left(\frac{i\Omega c}{a}\right) \left(\frac{\omega c}{a}\right)^{-i\frac{\Omega c}{a}} e^{-\frac{\pi \Omega c}{2a}}$$

$$S(\Omega) = |I|^2 = \frac{2\pi c}{\Omega a_0} \frac{1}{e^{\frac{2\pi \Omega c}{a_0}} - 1} \sim \underbrace{\frac{1}{e^{\frac{\hbar \Omega}{K_B T}} - 1}}_{\sim \text{Planck distribution}}$$

The observed frequency spectrum is that of a Planck distribution (black body spectrum) of temperature:

$$T = \frac{\hbar a_0}{2\pi K_B c}$$

Unruh temperature

$$\Rightarrow T = \frac{\hbar a_0}{2\pi K_B c}$$

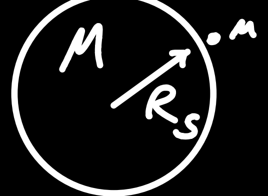
Temperature seen by observer with proper acceleration a_0

Recall Equivalence principle:

Effects of Gravity are indistinguishable (locally) from an acceleration.

What is local acceleration at Black holes horizon?

- Newtonian fudge (gives same result of more sophisticated GR calculation)



$$f = \frac{GMm}{R_s^2} \qquad a_0 = \frac{f}{m} = \frac{GM}{R_s^2}$$

$$R_s = \frac{2GM}{c^2}$$

$$a_0 = \frac{GM\,c^4}{4\,G^2M^2} = \frac{c^4}{4GM}$$

So we have the temperature of the black hole horizon:

$$\boxed{T_{B.H} = \frac{\hbar a_0}{2\pi K_B c} = \frac{\hbar c^3}{8\pi G M K_B}}$$

Using some classical thermodynamics:

$$dU = T\,dS$$

$$\frac{dU}{dS} = T$$

$$c^2 \frac{dM}{dS} = T$$

$$dS = \frac{c^2\, dM}{T}$$

$$dS = \frac{c^2 8\pi G M k_B}{\hbar c^3}\, dM$$

$$\Rightarrow S = \frac{4\pi G M^2 k_B}{\hbar c} = \frac{c^3 k_B 4\pi \left(\frac{2GM}{c^2}\right)^2}{4 G \hbar}$$

$$r_s = \frac{2GM}{c^2}$$

$$\Rightarrow S = \frac{c^3 k_B A}{4 G \hbar}$$

Horizon area

using $l_p = \sqrt{\frac{\hbar G}{c^3}}$ ← fundamental length scale of quantum gravity

Very neat result

$$\boxed{S = \frac{A}{4\, l_p^2}\, k_B}$$

Entropy of black hole, is it's area measured in units of the Planck length. (divided by 4)

Black holes no longer violate the 2nd law of thermodynamics. But loes more is still puzzling!...