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Mathematics for physics – Handout

<u>Vectors:</u>

Mathematical overview:

Out physical intuition of a vector is centered on the image of an arrow that points in the vectors direction and's length is representative of the vectors magnitude.

Before the equation of a vector can be written down, the coordinate system it is represented in must be specified. The most common Cartesian system used is the Cartesian coordinate system (but there are others!). (Cartesian coordinate system is just the fancy name for the familiar x,y,z grid)

A basis vector is a vector of unit magnitude (length = 1) directed along a given coordinate axis.

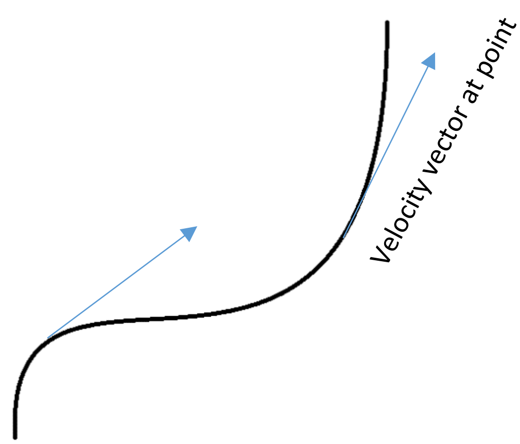
The $\vec{\imath}$ basis vector is directed along the x axis, $\vec{\jmath}$ along the y axis, $\vec{k}$ along the z axis.

Any vector in 3D space can be written as a sum of $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ vectors.

Hence; $\vec{a} = a_x\vec{\imath} + a_y\vec{\jmath} + a_y\vec{k}$ is the general form of a vector $\vec{a}$. a_x, a_y, a_z are the x,y and z components respectively.

Imagine building an x and y axis made from two perpendicular rods. Now take another rod representing the vector $\vec{a}$, placing one end at the origin, and pointing it in the appropriate direction. Imagine shining a torch on the $\vec{a}$ rod in the direction perpendicular to the x axis. The length of the shadow cast by the $\vec{a}$ rod along the x axis is the x component of $\vec{a}$ (a_x).

Another way of representing a 3D vector is by using column vector notation. Rather than writing; $\vec{a} = a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}$, we would write; $\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$



Consider the curve to the right. This represents the paths of a particle in a plane. The tangents to this curve at a point gives the direction of the particles velocity at this point.

Geometrically, adding two vectors involves taking one vector (remember the arrow image) and placing the other vectors un-pointed end at the first vectors point. By drawing a line between the first vectors un-pointed end, and the second vectors pointed end we get a geometric representation of the resultant vector.

Algebraically, adding two vectors is also very simple; $\vec{a} + \vec{b} = (a_x + a_x)\vec{\imath} + (a_y + b_y)\vec{\jmath} + (a_z + b_z)\vec{k}$

It is also useful to note... $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Two vectors $\vec{c}$ and $\vec{d}$ are only equal if all of their components are equal.

Physical application:

Many physical quantities are wonderfully represented as vectors. To name a few… Force, acceleration, velocity, displacement, electric and magnetic field strength, momentum, angular momentum, angular velocity, current, torque…

A vector equation is (rather unsurprisingly) an equation involving vectors.

For example, Newton's second law is most accurately written in vector form; $\vec{F} = m\vec{a}$. This tells us, that the direction of the net force pushing a particle and its acceleration are in the same direction (Note, mass is a scalar).

Writing out the force and acceleration vector components in full shows; $\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = m \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} ma_x \\ ma_y \\ ma_z \end{pmatrix}$

Rather than writing out the Newton's 2nd law thrice for each direction, we can nice and neatly write the law in its vector form.

<u>Calculus:</u>

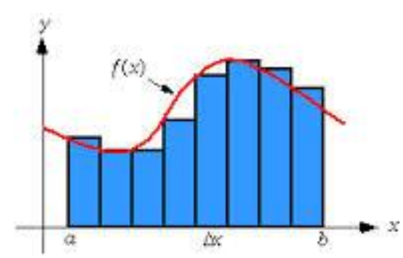
Mathematical overview:

As you have studied in C1 and C2 maths, differentiating a function gives a second function that outputs the gradient of the tangent to the first function at a given point. Integrating a function is the inverse of differentiating it, integrating gives a function that outputs the area between the $y = f(x)$ curve and the x axis.

$$\frac{d}{dx}(x^n) = nx^{n-1} \qquad \textit{the derivative}$$

$$\int x^n dx = \frac{1}{n+1}x^{n+1} \qquad \textit{the integral}$$

The integral the limit of a sum. Imagine we have a function y = f(x), and we wish to approximate the area underneath it. It order to do this, we could divide the area under the curve up into rectangles, and sum their areas as the image to the left illustrates.



The sum we have will look something like this;

$$A \sim \sum f(x_i)\Delta x_i$$

Here, A is the area under the curve, x_i is a point along the x axis (hence $f(x_i)$ is the height above that point), and Δx_i is the chosen width of the rectangle centered at x_i.

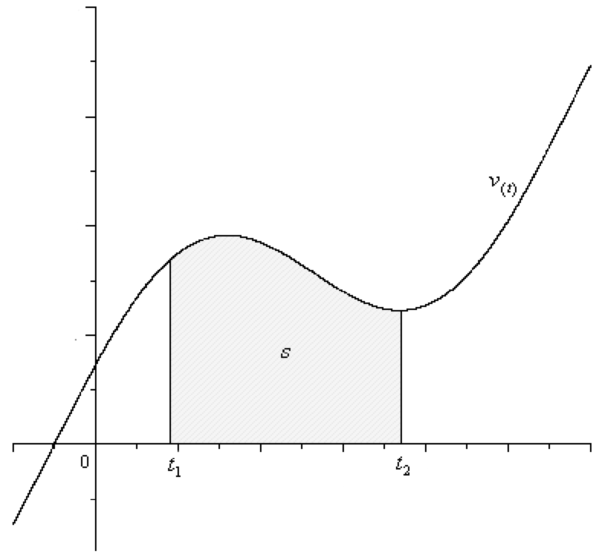
The smaller rectangles we use to approximate this area, the better they fit into the contours of the curve, hence the better the approximation for the area.

Suppose we let the width of the rectangles (Δx_i) become arbitrarily small (close to zero). If we performed this sum, the result would be exactly the area under the curve.

Mathematically, what this means is to let $\Delta x_i \to 0$. When we take this limit, the sum becomes an integral.

$$\lim_{\Delta x_i \to 0}\left(\sum f(x_i)\Delta x_i\right) = \int f(x)dx \qquad \textit{To use the formal notation}$$

Physical application:



As you know from your study of mechanics is physics, the area under a velocity-time graph gives the displacement. This should lead you to the conclusion that the integral the velocity curve gives displacement.

However you should also bear in mind the idea of the integral as a sum;

Consider a car that moves with a speed v_i at time t_i for a time Δt_i for any i. The total distance this car will travel is;
$s = v_1\Delta t_1 + v_2\Delta t_2 + v_3\Delta t_3 + \cdots = \sum(v_i\Delta t_i)$

If the cars speed changes continuously, i.e. its velocity-time curve is smooth, the Δt time intervals we sum over must be infinitely tiny. As the cars speed is never constant for more than an instant. Hence the distance travelled is given by the integral;

$$\Delta s = \int_{t_i}^{t_f} v(t)dt$$

You will also be aware, that the gradient of the tangent to a displacement- time graph is velocity at that point. This implies;

$$v = \frac{ds}{dt}$$

Kinematic Equations – General Calculus Form

$$a_x = \frac{dv_x}{dt}$$

$$v_{xf} - v_{xi} = \int_0^t a_x dt$$

$$v_x = \frac{dx}{dt}$$

$$x_f - x_i = \int_0^t v_x dt$$

Differential equations:

As the name would suggest, a differential equation is an equation involving a function $f(x)$ say, and its derivatives (along with other x's thrown in).

For example, $\frac{dy}{dx} = 2x$ is a simple DE. To solve a DE, is to find a function $y = f(x)$ that satisfies this equation. A solution to this DE is $y = x^2$.

A very useful method of solving certain DE's you will look at in C4 is the separation of variables technique. This involved bringing all the x's to one side multiplied by a dx, and all the y's to the other multiplied by a dy. Then integrating both sides. For example:

$$x\frac{dy}{dx} = \frac{x^4}{y} \to ydy = x^3dx \to \int ydy = \int x^3dx \to \frac{1}{2}y^2 = \frac{1}{4}x^4 + C \to y = \pm\sqrt{\frac{1}{2}x^4 + 2C}$$

Introduction to complex numbers: (Further reading)

Complex numbers are central to both physics and mathematics. They are immeasurably useful in solving differential equations, they work their way beautifully and unavoidably into quantum mechanics, and they have about them a superb mathematical beauty and structure.

Before discussing what a complex number is, it will be useful to recall what a real number is. The real numbers are all of the numbers that lie on the number line stretching from minus infinity to positive infinity. They of course include both the rational and irrational numbers. A complex number however, does not exist on this number line, which may initially be cause for some concern… But bear with me.

Let us consider the quadratic equation, $x^2 + 3x + 2 = 0$. This equation can quite easily be factorized, and found to have solutions of $x = 1$ and $x = 2$. What about the equation $x^2 + 3x + 3 = 0$? We cannot immediately factor this equation, so we try the quadratic formula, which yields; $x = \frac{1}{2}\left(-3 \pm \sqrt{-3}\right)$. At this point, we would probably stop, go back, and write under the equation 'no solution'. Which is perfectly true for the real numbers, this quadratic does indeed have no solution that is a real number. Suppose we just want to have a bit of fun though, so we keep playing... Let's rewrite our 'solution' slightly; $x = \frac{1}{2}\left(-3 \pm \sqrt{3}\sqrt{-1}\right)$. Now suppose, for a laugh, we decide to give a definition to $\sqrt{-1}$, we know it isn't a real number (we can't point to it on a number line) but we're mathematicians, we do what we want. Let's define it as such; $i = \sqrt{-1}$, 'i' is called the 'imaginary unit' (hence $i^2 = -1$). Now at this point, we have no idea if this makes any sense, whether it fits into the pre-existing frame work of mathematics, or whether it has any 'real' meaning, all we have done is defined a shorthand such that we can write our solution as $x = \frac{1}{2}\left(-3 \pm i\sqrt{3}\right)$.

The rather magical thing is, when we define the square root of minus one, to be this 'imagionary' number i, we do not damage the structure of mathematics. All of our previous maths remains true, and in fact we add new structure, and widen the scope of our mathematics! This is truly remarkable.

Let us write something similar in form to our solution discussed above. Consider the expression; $z = a + ib$, this is the general form of a complex number 'z'. Our complex number has a real and an imagionary component, a and b respectively (Note, a and b themselves are real numbers, ib is an imagionary number).

The algebra of complex numbers works similarly to the algebra of real numbers, the difference being when we add to complex numbers, we add their real components and their imagionary components separately. i.e. $(a + ib) + (c + id) = (a + c) + i(b + d)$. Multiplication works in a similar way to brackets of real numbers; $(a + ib)(c + id) = ac + iad + ibc + (ib)(id) = (ac - bd) + i(ad + bc)$.

This form of a general complex number lends itself very well to a geometric interpretation, much like how the reals can be thought of as a number line, the complex numbers can be thought of as a plane. This plane is often called the complex plane, or the Argand plane. For $z = a + ib$ we simply define as point in this plane as being a point with coordinates (a, b), so we plot the real component along the x axis, and the imagionary component along the y axis.

I had intended here to write about why the term 'imaginary number' is rather foolish, complex numbers are as 'real' as any number. However, I thought it better to use this space to recommend to you all to take a look at the chapters in the FP1 and FP2 textbooks on complex numbers, they really are important.