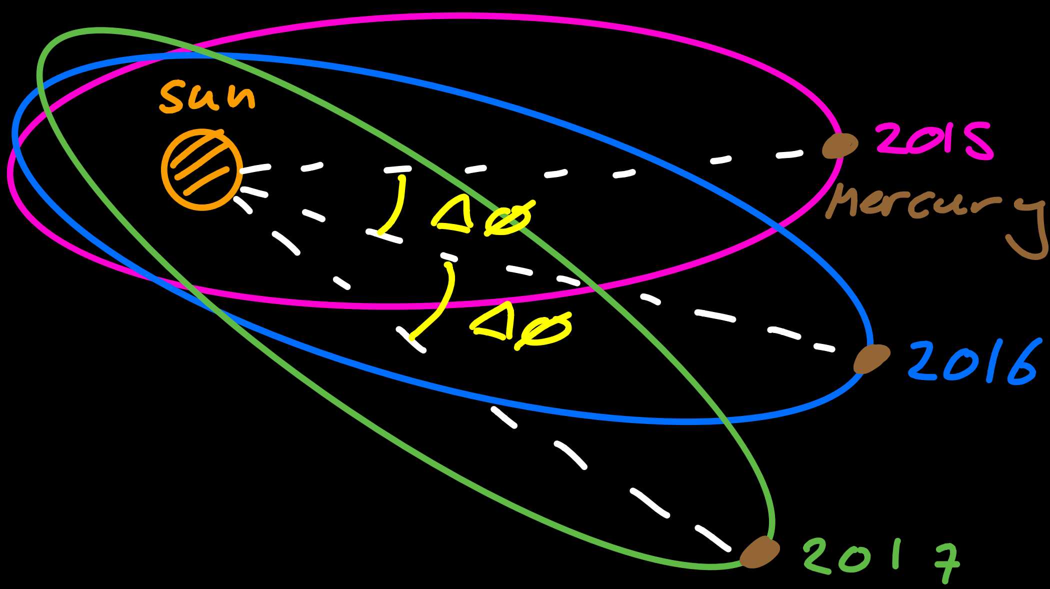
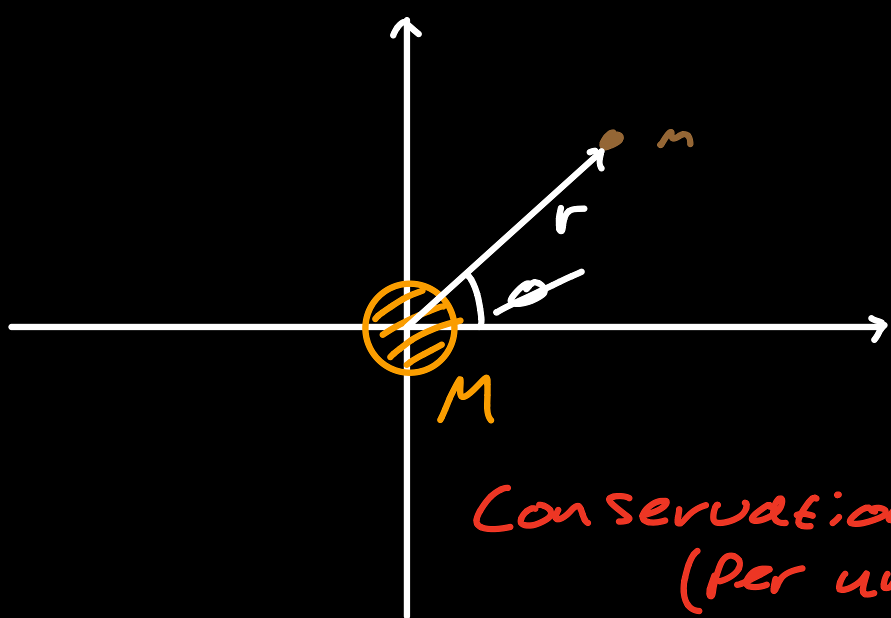
Mercury's Orbital Precession



• Newtonian Orbital Mechanics



Conservation of Energy
(Per unit Mass)

$$E = \frac{1}{2}|\underline{v}|^2 - \frac{GM}{r}$$

In Polar coords;

$$|\underline{v}|^2 = \dot{r}^2 + r^2\dot{\phi}^2$$

where $\dot{r} \equiv \frac{dr}{dt}$, $\dot{\phi} \equiv \frac{d\phi}{dt}$

$$E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 - \frac{GM}{r} \qquad (1)$$

Conservation of angular momentum;
(Per unit mass)

$$J = r^2\dot{\phi}$$

$$\dot{\phi} = \frac{J}{r^2} \qquad (2)$$

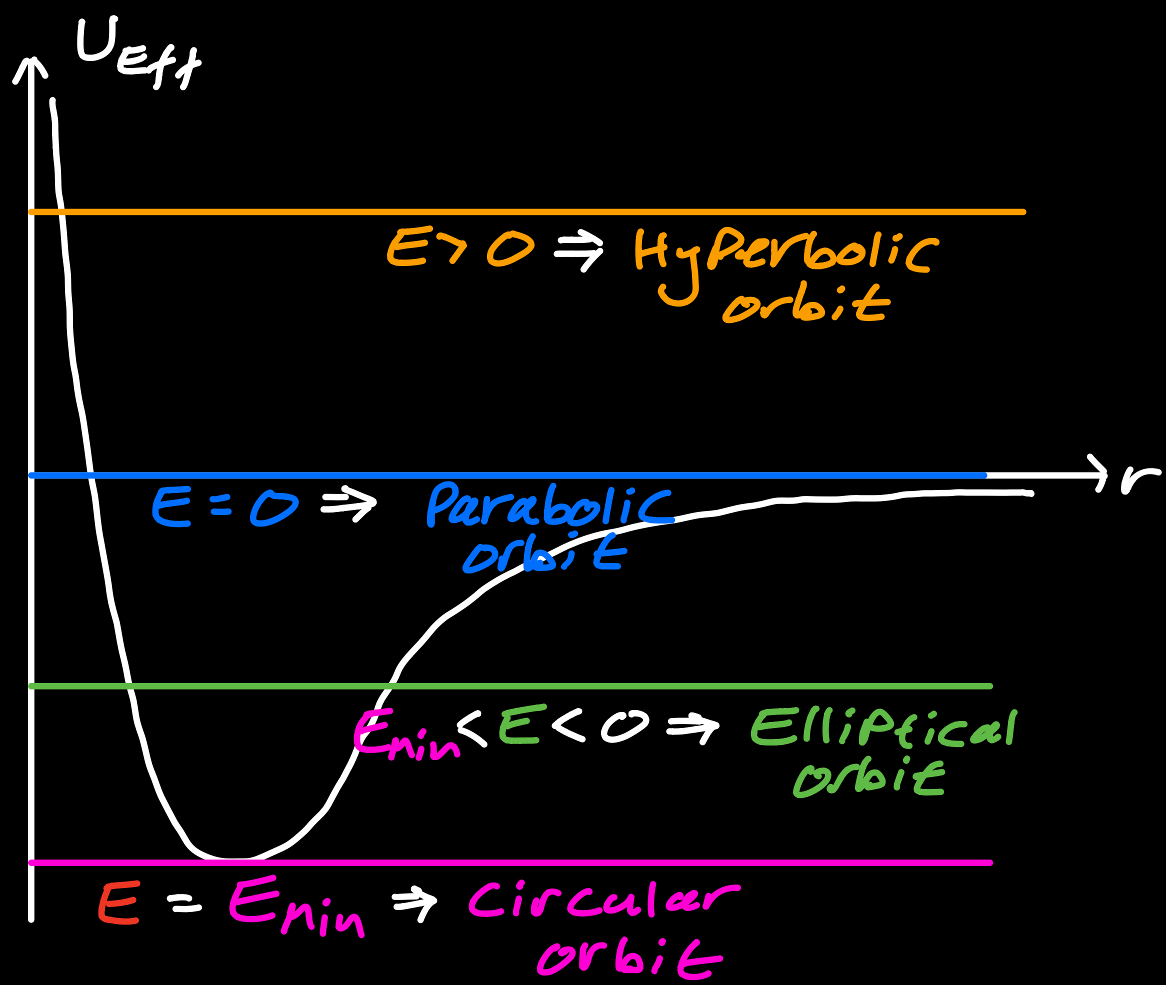
Sub (2) into (1);

$$\boxed{E = \frac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2} - \frac{GM}{r}}$$

$$E = \frac{1}{2}\dot{r}^2 + U_{Eff}(r)$$

1D Problem

$$U_{Eff}(r) = \frac{J^2}{2r^2} - \frac{GM}{r}$$



• Einsteinian Orbital Mechanics

Schwarzschild Metric ;

$$ds^2 = -c^2 dt^2\left(1 - \frac{2GM}{c^2 r}\right) + \ldots$$

$$\ldots + dr^2\left(1 - \frac{2GM}{c^2 r}\right)^{-1} + r^2 d\Omega^2 = -c^2 d\tau^2$$

Proper time along geodesic

with: $d\Omega^2 = d\theta^2 + r^2 \sin^2\theta \, d\phi^2$

We fix $\theta = \frac{\pi}{2}$ as our orbital plane $\Rightarrow d\Omega^2 = d\phi^2$

$$ds^2 = -c^2 dt^2\left(1 - \frac{2GM}{c^2 r}\right) + dr^2\left(1 - \frac{2GM}{c^2 r}\right)^{-1} + \ldots$$

$$\ldots + r^2 d\phi^2$$

$$L = \left(\frac{ds}{d\tau}\right)^2 = -c^2$$

$$= -c^2(\dot{t})^2\left(1 - \frac{2GM}{c^2 r}\right) + (\dot{r})^2\left(1 - \frac{2GM}{c^2 r}\right)^{-1} + \ldots$$

$$\ldots + r^2\dot{\phi}^2$$

Euler-Lagrange equations:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = \frac{\partial L}{\partial x^\mu}$$

$$x^\mu = t, r, \phi$$

- ϕ:

$$\frac{\partial L}{\partial \phi} = 0$$

$$\Rightarrow \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{Constant}$$

$$\frac{\partial L}{\partial \dot{\phi}} = 2r^2\dot{\phi} = 2J \leftarrow$$

Angular momentum per unit mass (in analogy to Newtonian case)

$$\Rightarrow \boxed{\dot{\phi} = \frac{J}{r^2}}$$

• t:

$$\frac{\partial L}{\partial t} = 0$$

$$\Rightarrow \frac{\partial L}{\partial \dot{t}} = \text{constant}$$

$$\frac{\partial L}{\partial \dot{t}} = -2c^2\dot{t}\left(1 - \frac{2GM}{c^2 r}\right) = 2E$$

Energy per unit mass

$$\Rightarrow \boxed{\dot{t} = -\frac{E}{c^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}}$$

$$-c^2 = -c^2(\dot{t})^2\left(1 - \frac{2GM}{c^2 r}\right)$$

$$+(\dot{r})^2\left(1 - \frac{2GM}{c^2 r}\right)^{-1} + r^2\dot{\phi}^2$$

Sub in boxed expressions for $\dot{\phi}$ and $\dot{t}$

$$-c^2 = -c^2\left[\frac{E}{c^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}\right]^2\left(1 - \frac{2GM}{c^2 r}\right) + \dots$$

$$\dots + \dot{r}^2\left(1 - \frac{2GM}{c^2 r}\right)^{-1} + \frac{J^2}{r^2}$$

$$-c^2 = -\left(\frac{E}{c}\right)^2\left(1-\frac{2GM}{c^2 r}\right)^{-1} + \ldots$$

$$\ldots + \dot{r}^2\left(1-\frac{2GM}{c^2 r}\right)^{-1} + \frac{J^2}{r^2}$$

Multiply equation by $\frac{1}{2}\left(1-\frac{2GM}{c^2 r}\right)$

$$\frac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2}\left(1-\frac{2GM}{c^2 r}\right) + \ldots$$

$$\ldots + c^2\frac{1}{2}\left(1-\frac{2GM}{c^2 r}\right) = \frac{1}{2}\left(\frac{E}{c}\right)^2$$

$$\boxed{\frac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2} - \frac{GM}{r} - \frac{GMJ^2}{c^2 r^3} = \frac{1}{2}c^2\left[\left(\frac{E}{c^2}\right)^2 - 1\right]}$$

Compare with Newtonian equation

$$\boxed{\frac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2} - \frac{GM}{r} = E}$$

$$U_{Eff}(r) = \frac{J^2}{2r^2} - \frac{GM}{r} - \underbrace{\frac{GMJ^2}{c^2 r^3}}$$

General relativistic correction

Change variables:

$$u = \frac{1}{r}$$

$$\frac{du}{d\phi} = -\frac{1}{r^2}\dot{r}\frac{1}{\dot{\phi}} = -\frac{\dot{r}}{J}$$

$$\Rightarrow \quad \dot{r} = -J\frac{du}{d\phi}$$

$\Rightarrow$ Equation becomes:

$$+\frac{J^2}{2r^2} - \frac{GM}{r} - \frac{GMJ^2}{c^2r^3} = \frac{1}{2}c^2\left[\left(\frac{E}{c^2}\right)^2 - 1\right]$$

$$\frac{J^2}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{J^2}{2}u^2 - GMu + \ldots$$

$$\ldots - \frac{GMJ^2}{c^2}u^3 = \text{Constant}$$

Differentiate w.r.t ϕ

$$J^2 \left(\cancel{\frac{du}{d\phi}}\right)\left(\frac{d^2u}{d\phi^2}\right) + J^2 u \cancel{\frac{du}{d\phi}} - GM \cancel{\frac{du}{d\phi}} + \ldots$$

$$\ldots - \frac{3GMJ^2}{c^2} u^2 \cancel{\frac{du}{d\phi}} = 0$$

$$\frac{d^2u}{d\phi^2} + u - \frac{3GM}{c^2}u^2 - \frac{GM}{J^2} = 0$$

Consider small departures from circular orbit:

$$u = u_{circ} + \varepsilon f(\phi)$$

For a circular orbit, $\frac{d^2u}{d\phi^2} = 0$

$$\Rightarrow \boxed{u_{circ} - \frac{3GM}{c^2}u_{circ}^2 - \frac{GM}{J^2} = 0}$$

$$\varepsilon f''(\phi) + u_{circ} + \varepsilon f(\phi) + \ldots$$

$$\ldots - \frac{3GM}{c^2}\left[u_{circ} + \varepsilon f(\phi)\right]^2 - \frac{GM}{J^2} = 0$$

$$\varepsilon f''(\phi) + \varepsilon f(\phi) - \varepsilon \frac{6GM}{c^2} u_{circ} f(\phi) + \ldots$$

$$\ldots \cancel{+ u_{circ} - \frac{3GM}{c^2} u_{circ} - \frac{GM}{J^2}} + \ldots$$

from boxed equation above

$$\ldots - \frac{3GM}{c^2}\varepsilon^2 f(\phi) = 0$$

$$\cancel{\varepsilon} f''(\phi) + \cancel{\varepsilon} f(\phi)\left[1 - \frac{6GM}{c^2} u_{circ}\right] + \ldots$$

$$- \frac{3GM}{c^2}\varepsilon^{\cancel{2}} f(\phi) = 0$$

To leading order in ε

$$f''(\phi) + f(\phi)\left[1 - \frac{6GM}{c^2}u_{circ}\right] = 0$$

$$u_{circ} = \frac{1}{R}$$

Radius of circular orbit

$$f''(\phi) + f(\phi)\left[1 - \frac{6GM}{c^2 R}\right] = 0$$

$f(\phi)$ has oscillatory solutions

$$f(\phi) \sim e^{i\omega\phi}$$

$$-\omega^2 + \left[1 - \frac{6GM}{c^2 R}\right] = 0$$

$$\omega = \left(1 - \frac{6GM}{c^2 R}\right)^{\frac{1}{2}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\left(1 - \frac{6GM}{c^2R}\right)^{-\frac{1}{2}}$$

$$\simeq 2\pi + \frac{6\pi GM}{c^2R}$$

$$\boxed{\Delta\phi = \frac{6GM}{c^2R}}$$

- **Comparing with observation**

Measured precession	$\rightarrow \Delta\phi = 43$ arcseconds per century

$$R \sim 58\times10^{6}\ \text{KM}$$

$$M = 1.99\times10^{30}\ \text{kg}$$

$$\Delta\phi = \frac{6\pi\times6.67\times10^{-11}\times1.99\times10^{30}}{(3\times10^{8})^{2}\times5.55\times10^{10}} \simeq 5.01\times10^{-7}$$

radians per orbit

Mercury's orbital period = 87.97 days

$\Rightarrow$ Mercury has $\frac{365.25}{87.97} \times 100 = 415.20$ orbits per century

$\Rightarrow$ Calculated angle of precession per century $= 415.20 \times 1.53 \times 10^{-7}$

$= 6.35 \times 10^{-5}$ radians per century

π radians $= 648000$ arcseconds

$$\boxed{\text{Calculated precession} \longrightarrow 42.9 \text{ arcseconds per century}}$$