What is a tensor?

- Linear algebra perspective:

first need to understand 'what is a vector'.

A 'vector' is formally an element of a vector space → an algebraic structure that describes and generalizes the properties of vectors we are used to.

$$\underline{v} \in V_n(\mathbb{R})$$

(for vectors with real components)

we will picture these as n-dimensional column vectors

This is notation for an n-dimensional vector space over the real numbers

without stating axioms (you can find the vector space axioms in any textbook). $V_n(\mathbb{R})$ has important properties such as, linearity, closure, and multiplication by scalars.

We have also a different way of writing vectors:

$$\underline{v}^T \in \mathcal{V}_n^*(\mathbb{R})$$

We will picture these as row vectors

This notation is for an n-dimensional 'dual' vector space, over the real numbers.

'Dual' is nothing mysterious here. $\mathcal{V}_n^*(\mathbb{R})$ is just as much a vector space as $\mathcal{V}_n(\mathbb{R})$. Which one is 'dual' is really a matter of perspective.

i.e. $\mathcal{V}_n^{**}(\mathbb{R}) \cong \mathcal{V}_n(\mathbb{R})$

Formally, a dual vector space is the space of linear maps that take elements of the vector space to the real numbers.

We can understand $\underline{v}^T$ and $\underline{v}$ as being dual to one another through matrix multiplication

$$\underline{v}^T : \underline{v} \longrightarrow \mathbb{R}$$

through $$\underline{v}^T \underline{v} = |\underline{v}|^2 \in \mathbb{R}$$

So there are two ways of thinking about vectors, as elements of a vector space, or as elements of the corresponding dual vector space.

Note: we may just as we state this in reverse, then $V_n(\mathbb{R})$ is the dual space to $V_n^*(\mathbb{R})$ as

$$\underline{v} : \underline{v}^T \rightarrow \mathbb{R}$$

through $$\underline{v}^T \underline{v} = |\underline{v}|^2 \in \mathbb{R}$$

We may expand $\underline{v} \in V_n(\mathbb{R})$ and $\underline{v}^T \in V_n^*(\mathbb{R})$ in a component-basis form, as we are used to doing for column and row vectors.

$\underline{v} \in V_n(\mathbb{R})$ will be expanded over a basis of unit column vectors $\left\{ \underset{i=1}{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}, \underset{i=2}{\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}}, \ldots, \underset{i=n}{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}} \right\}$

we will denote these basis vectors as $\underline{e}_i$

and the corresponding components of $\underline{v}$ as v^i.

Such that:

$$\underline{v} = \sum_{i=1}^{n} v^i \underline{e}_i = \underline{v^i \underline{e}_i}$$

making use of the 'Einstein summation convention'

Likewise for $\underline{v}^T \in V_n^*(\mathbb{R})$ we will write

$$\underline{v}^T = \sum_{i=1}^{n} v_i \underline{e}^i = v_i \underline{e}^i$$

where $\underline{e}^i \in \{(1,0,\ldots,0), (0,1,\ldots,0), \ldots, (0,0,\ldots,1)\}$

from these two structures $\mathcal{V}_n(\mathbb{R})$ and $\mathcal{V}_n^*(\mathbb{R})$, we can construct more complicated objects.

Say we take two copies of the space $\mathcal{V}_n(\mathbb{R})$, and build a new object, made from a pair of vectors

$$\left.\begin{aligned} \underline{v}^{(1)} &\in \mathcal{V}_n^{(1)}(\mathbb{R}) \\ \underline{v}^{(2)} &\in \mathcal{V}_n^{(2)}(\mathbb{R}) \end{aligned}\right\}$$

where the $^{(1)}$ and $^{(2)}$ upper scripts are just labels for each copy of $\mathcal{V}_n(\mathbb{R})$

$$\underbrace{\left(\underline{v}^{(1)}, \underline{v}^{(2)}\right)} \in \mathcal{V}_n^{(1)} \otimes \mathcal{V}_n^{(2)}$$

This is a tensor product symbol

Really, it is defined by what we're doing on the left hand side. 'stacking' copies of $\mathcal{V}_n$.

often this is just written as:

$$\mathcal{V}_n \otimes \mathcal{V}_n$$ ← This is a tensor space over the real numbers.

Note: $\dim(\mathcal{V}_n \otimes \mathcal{V}_n) = n^2$

The 'rank' of a tensor space tells us how many copies of $\mathcal{V}_n$ or $\mathcal{V}_n^*$ we are stacking.

E.g. If we stack p-copies of $V_n(\mathbb{R})$ and q-copies of $\mathcal{V}_n^*(\mathbb{R})$, this is denoted a rank (p,q)-tensor.

$$\underbrace{\mathcal{V}_n \otimes \mathcal{V}_n \otimes \ldots \otimes \mathcal{V}_n}_{p\text{-copies}} \otimes \underbrace{\mathcal{V}_n^* \otimes \mathcal{V}_n^* \otimes \ldots \otimes \mathcal{V}_n^*}_{q\text{-copies}}$$

Note: $\dim(\text{*above*}) = n^{p+q}$

for $V_n(\mathbb{R})$ corresponding to the space of all real n-dimensional vectors (as we have restricted ourselves to from the off);

$$\begin{tikzcd} V \arrow[r, bend left, "\text{dual}"] & V^* \arrow[l, bend left, "\text{dual}"] \end{tikzcd}$$

$V_n^{(1)} \otimes V_n^{(2)}$ can be understood as an $n \times n$ Matrix

which will map two copies of V_n^* to the real numbers.

$$(\underline{v}_n^{(1)}, \underline{v}_n^{(2)}) : (\underline{v}_n^{T(1)}, \underline{v}_n^{T(2)}) \longrightarrow \mathbb{R}$$

through
$$\underline{v}_n^{T(1)} \underline{v}_n^{(1)} + \underline{v}_n^{T(2)} \underline{v}_n^{(2)} \in \mathbb{R}$$

$V_n^* \otimes V_n$ can also be understood as an $n \times n$ Matrix, but with different components

• Special Relativity Perspective:

Principal of special relativity: The laws of physics are the same in all inertial reference frames.

To switch between inertial frames, we have Lorentz transformations. for measurements of times and lengths, Lorentz transformations preserve the invariant interval:

$$S^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

Transformations that preserve this interval have structure of a group.

$SO(1,3) \leftarrow$ The Lorentz group.

A 'group' is an abstract structure. It abstractly describes the symmetry of the system, but it doesn't explicitly tell us how physical quantities should transform.

group $\rightleftharpoons$ symmetry

Physical quantities are built out of vector spaces (or a scalars).

How is a symmetry described by a group realized on a vector space?

The answer is, through a 'group representation'.

A group representation is a linear map from each group element to an invertible matrix.

For a physical quantity represented by $V_4(\mathbb{R})$ (e.g. 4-velocity) $\underline{v} \in V_4(\mathbb{R})$ will transform as a representation;

$$R_v : g \in SO(1,3) \longrightarrow \underline{\underline{R}}_v \in GL(V_4)$$

$$\underline{\underline{R}}_V \in \overbrace{GL(V_4)}^{\text{4x4 invertible matrices}}$$

$\uparrow$ 'general linear group'

$$\underline{V}' = \underline{\underline{R}}_V \underline{V}$$

for a physical quantity represented by $V_4 \otimes V_4$ (e.g. the stress energy tensor) $(U^{(1)}, U^{(2)}) \in V_4 \otimes V_4$ will transform as a representation;

$$R_{V \otimes V} : g \in SO(1,3) \to \underline{\underline{R}}_V (\circ) \underline{\underline{R}}_V^{-1} \in GL(V_4 \otimes V_4)$$

Putting this into standard SR notation:

$$x^\mu \in V_4(\mathbb{R}) \quad \text{'contravariant component'}$$

$$x_\mu \in V_4^*(\mathbb{R}) \quad \text{'covariant component'}$$

$$\underline{\underline{R}}_V = \Lambda^\mu{}_\nu \in GL(V_4)$$

$$\underline{\underline{R}}_{V^*} = \Lambda_\mu{}^\nu \in GL(V_4^*)$$

If we build physics equations out of these representations of $SO(1,3)$:

our equations will transform in a reliable way between reference frames, whereby their form will not change.

E.g.

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu - P \eta_{\mu\nu}$$ ← Stress energy tensor for perfect fluid in some reference frame S_1.

$$T_{\mu\nu} \to \underset{\uparrow}{\tilde{T}_{\mu\nu}} = \underbrace{(\tilde{\rho} + \tilde{P}) \tilde{u}_\mu \tilde{u}_\nu - \tilde{P} \eta_{\mu\nu}}_{\text{same form as before}}$$

In frame S_2

$$e^{iq\alpha}$$

$$e^{iq\alpha(x)}$$