17. A parachutist jumps out of a plane at height h. She is subject to air resistance with a force of $-\alpha v^2$. The equation of her motion is given by

$$m\frac{dv}{dt} = mg - \alpha v^2.$$

a) What are the units of α?

b) Calculate the terminal velocity of the parachutist.

c) Estimate how much work is done by the air resistance as she falls, assuming that she is falling at near terminal velocity by the time she reaches the ground. [7]

a)

$$[mg] = kg\, m s^{-2}$$

$$[\alpha v^2] = kg\, m s^{-2}$$

$$[\alpha](m \cancel{s^{-1}})^{\cancel{2}\,2} = kg\, \cancel{m}\cancel{s^{-2}}$$

$$[\alpha] = kg\, m^{-1}$$

b)

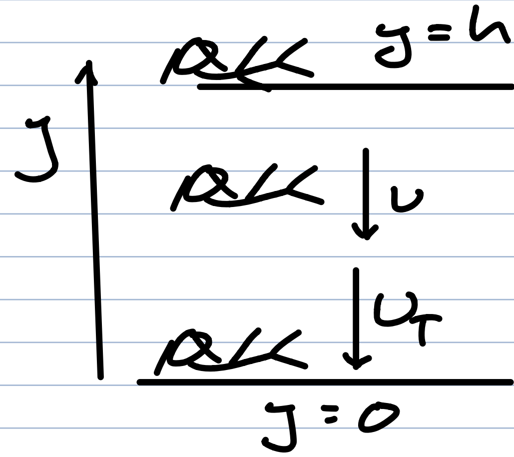
$$\frac{dv}{dt} = 0$$

$$\underbrace{m\frac{dv}{dt}}_{=0} = mg - \alpha v^2$$

$$mg - \alpha v^2 = 0$$

$$U_T = \sqrt{\frac{mg}{\alpha}}$$

c) $$m\frac{dU}{dt} = mg - \alpha U^2$$



$$E_T(y=h) = \frac{1}{2}m(0)^2 + mgh = mgh$$

$$E_T(y=0) = \frac{1}{2}mU_T^2 + mg(0) = \frac{1}{2}mU_T^2$$

$$E_T(y=h) - W = E_T(y=0)$$

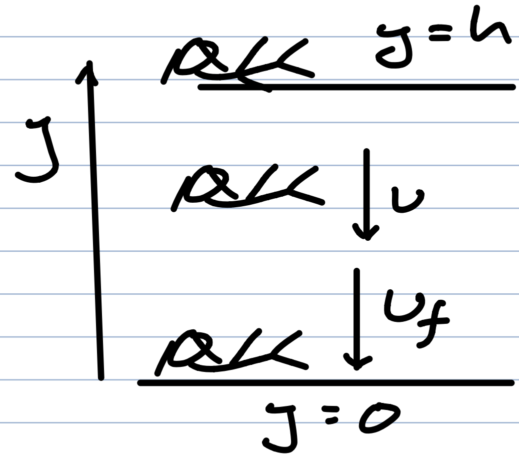
$$W = E_T(y=h) - E_T(y=0)$$

$$W = mgh - \frac{1}{2}mU_T^2$$

$$U_T^2 = \frac{mg}{\alpha}$$

$$W = mg\left(h - \frac{1}{2\alpha}\right)$$

Analytically:



$$m \frac{dv}{dt} = mg - \alpha v^2$$

$$W = mgh - \frac{1}{2} m v_f^2$$

$y = h,\ v = 0$

$y = 0,\ v = v_f$

$$\frac{dv}{dt} = \frac{dv}{dy} \underbrace{\frac{dy}{dt}}$$

$$\frac{dy}{dt} = -v$$

$$\frac{dv}{dt} = -v \frac{dv}{dy}$$

$$-mv \frac{dv}{dy} = mg - \alpha v^2$$

$\div mg$

$$-\frac{1}{g} v \frac{dv}{dy} = 1 - \frac{\alpha}{mg} v^2$$

$$\int_0^{v_f} \frac{v\,dv}{1 - \frac{\alpha}{Mg} v^2} = \int_h^0 -g\,dy \qquad \begin{matrix} y = h,\ v = 0 \\ y = 0,\ v = v_f \end{matrix}$$

$$-\frac{Mg}{2\alpha} \ln \left| 1 - \frac{\alpha}{Mg} v^2 \right| \Bigg|_0^{v_f} = -gy \Big]_h^0$$

$$-\frac{Mg}{2\alpha} \ln \left| 1 - \frac{\alpha}{Mg} v_f^2 \right| = gh$$

$$1 - \frac{\alpha}{Mg} v_f^2 = e^{-\frac{2\alpha h}{M}}$$

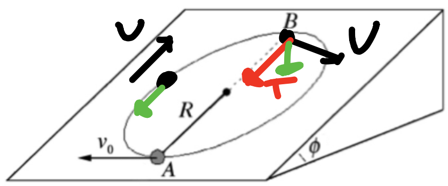
$$v_f = \sqrt{\frac{Mg}{\alpha}} \left[1 - e^{-\frac{2\alpha h}{M}} \right]^{\frac{1}{2}}$$

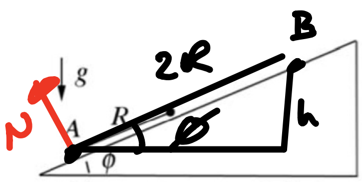
$$v_f \sim \sqrt{\frac{Mg}{\alpha}} \quad \text{when} \quad \underbrace{e^{-\frac{2\alpha h}{M}} << 1}$$

$$v_f = v_T \left[1 - e^{-\frac{2\alpha h}{M}} \right] \qquad \begin{matrix} \frac{2\alpha h}{M} >> 0 \\ h >> 0 \end{matrix}$$

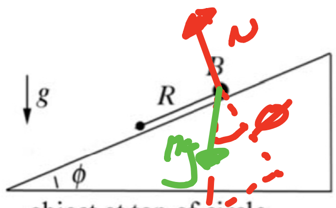
Next Q:

3. **Inclined plane** A body of mass m is attached to one end of a string of length R. The other end of the string is fixed on an inclined plane making an angle ϕ with the horizontal as shown in the figure. The body has speed v_0 at the bottom of the circle (point A). The body undergoes circular motion. There is a coefficient of sliding friction μ between the body and the plane. The downward acceleration of gravity is g. Express all answers in terms of m, ϕ, v_0, g, μ and R as needed.





object at bottom of circle
(side view)



object at top of circle
(side view)

(a) How much work does the friction force do on the body as it moves from the bottom of the circle (point A) to the top of the circle (point B)?

(b) What is the tension in the string when it reaches point B? Express your answer in terms of m, ϕ, v_0, g, μ and R as needed.

a) $N = mg\cos\phi$

$f_r = N\mu$

$W = N\mu\pi R$

$W = mg\mu\pi R\cos\phi$

b) Resolving down the slope

$$\boxed{T + mg\sin\phi = \frac{mv^2}{R}}$$

$$E_T(A) = \frac{1}{2}mv_0^2$$

$$E_T(B) = \frac{1}{2}mv^2 + mg\,2R\sin\phi$$

$$E_T(A) - W = E_T(B)$$

$$E_T(A) - mg\mu\pi R\cos\phi = E_T(B)$$

$$\boxed{\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 - mg\mu\pi R\cos\phi - 2mgR\sin\phi}$$

$$T = \frac{mv^2}{R} - mg\sin\phi$$

$$T = \frac{mv_0^2}{R} - 2mg\mu\pi\cos\phi - 4mg\sin\phi - mg\sin\phi$$

$$T = \frac{mv_0^2}{R} - mg\left[5\sin\phi + 2\pi\mu\cos\phi\right]$$