White Dwarf Stars & Quantum Mechanics

Schrödinger equation (1D) : $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi = i\hbar\frac{\partial\psi}{\partial t}$

Two kinds of particle $\longrightarrow$ fermions (odd half integer spin), Bosons (integer spin)

$\psi(x_1, x_2, x_3, \ldots, x_n) \leftarrow$ wave function for n particles at positions $x_1, \ldots, x_n$

fermions : $\psi_f(x_1, x_2) = -\psi_f(x_2, x_1)$

Bosons : $\psi_B(x_1, x_2) = \psi_B(x_2, x_1)$

Probability of finding Particle 1 at x_1 & Particle 2 at x_2 $= \int_{-\infty}^{\infty} \psi(x_1, x_2)\, \psi^*(x_1, x_2)\, dx$

If $x_1 = x_2$

$$\psi_f(x_1, x_2) = -\psi_f(x_2, x_1)$$

$$\rightarrow \quad \psi_f(x_1, x_1) = -\psi_f(x_1, x_1)$$

$$\Rightarrow \quad \psi_f(x_1, x_1) = 0$$

$\longrightarrow$ Pauli exclusion principle.

Can't cram two electrons in a system into the same state.

This principle is what keeps white dwarfs from collapsing. However, there is a fundamental limit to how massive a white dwarf can be.

In order to work out this limit (the Chandrasehkar mass), we will count the number of allowed states we can cram into a given volume.

Time independent Schrödinger equation for free particle in a 1D box $[0, L]$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

$$\Rightarrow \quad \psi(x) = A\sin(kx) + B\cos(kx)$$

with $k = \frac{\sqrt{2mE}}{\hbar}$

B.C's $\quad \psi(x=0) = \psi(x=L) = 0$

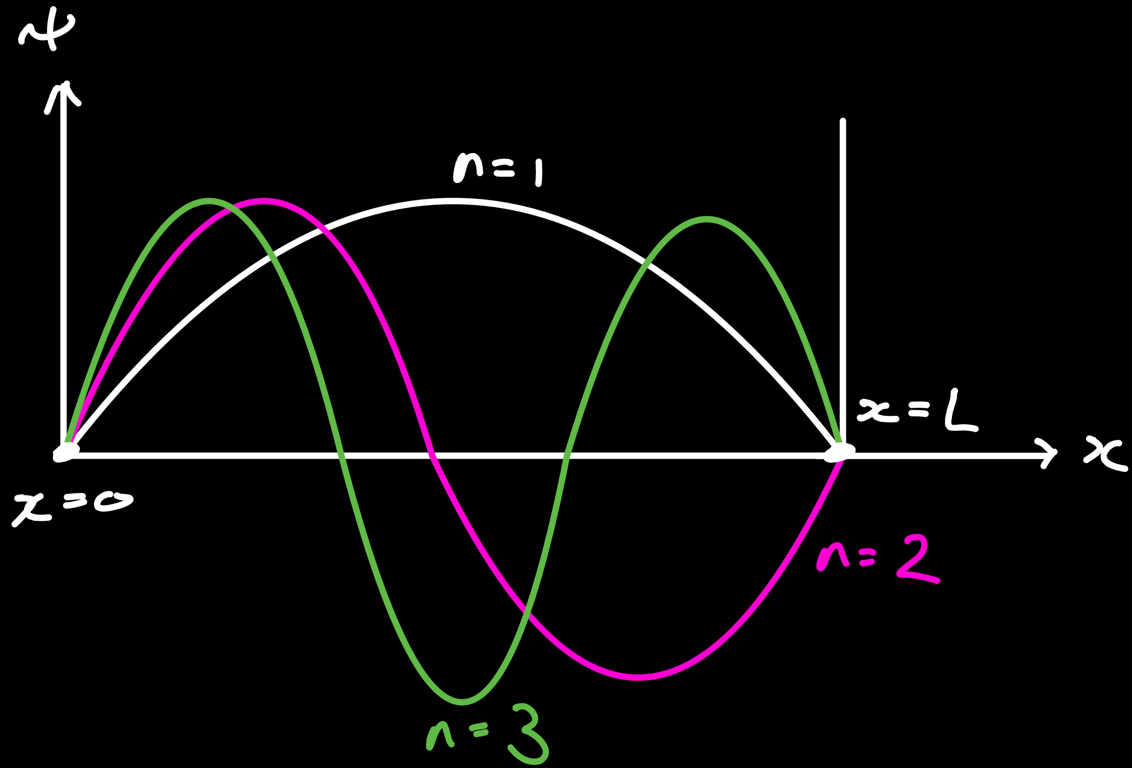
$$kL = n\pi$$

for $n \in \mathbb{Z}$

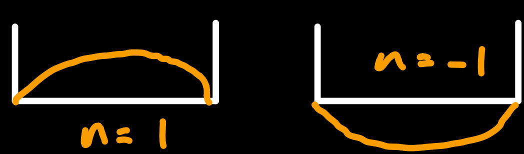
$$\Rightarrow \psi(x) = \underset{\uparrow}{A} \sin\left(\frac{n\pi}{L}x\right)$$

normalization factor (unimportant for us)

→ ∞ number of allowed states with momentum $p^{(n)} = \frac{\hbar n \pi}{L}$



Only interested in states with $n > 0$, as $n < 0$ will be the same wave function, but flipped. e.g.



$$|p^{(n)}| = \frac{\hbar n \pi}{L}$$

We will count the number of allowed states using the momentum.

E.g. for $|p| \leq \frac{6\hbar\pi}{L}$, there are 6 allowed momentum states $n = 1, 2, 3, 4, 5, 6$

What if we want to know the number of states between the momenta p and $p + dp$

factor of a half, as we are only looking for states with $p > 0$

$$n = \frac{1}{2}\frac{L}{\pi\hbar}\, p$$

$$dn = \frac{1}{2}\frac{L}{\pi\hbar}\, dp$$

This expression gives the number of states between $|p|$ and $|p| + d|p|$

We can generalize this to a particle in 3D:

$$dn = \left(\frac{1}{2}\frac{L}{\pi}\right)^3 dp_x\, dp_y\, dp_z$$

$\uparrow$

Number of states between $|\underline{p}|$ and $|\underline{p}| + d|\underline{p}|$

When there is no prefered momentum direction;

$$dp_x\, dp_y\, dp_z = 4\pi p^2\, dp$$

$$(\text{where } p = |\underline{p}|)$$

$$\Rightarrow \quad dn = \left(\frac{L}{2\pi\hbar}\right)^3 4\pi p^2\, dp$$

But we've missed something. Electrons have spin. This means we have an additional degree of freedom, freeing up some (phase) space to cram electrons into. As there are two possible spin states for each electron, we can fit twice as many states as we've previously figured out into a given momentum interval.

$$\boxed{dn = \frac{V}{\pi^2 \hbar^3} p^2 dp}$$

where $V = L^3$ is the volume of our 3D box.

If we assume that the pressure in white dwarfs is largely due to a degenerate electron gas, we can calculate this pressure under the assumption that the gas is fully degenerate.

Degenerate → Every state up to some upper limit is occupied.

first, let's count the total number of electrons in terms of the filled momentum states:

$$N_e = \int_0^{p_f} \frac{V}{\pi^2 \hbar^3} p^2 dp = \frac{1}{3} \frac{V}{\pi^2 \hbar^3} p_f^3$$

'fermi-momentum'
The maximum filled momentum state.

$$\boxed{n_e = \frac{N_e}{V} = \frac{1}{3} \frac{1}{\pi^2 \hbar^3} p_f^3}$$

Electron number density

Now, lets look for the pressure of the electron gas:

$$\underbrace{P}_{\substack{\text{pressure} \\ = \text{Average momentum flux}}} = \int_0^{p_F} dp\, p^2 \underbrace{\frac{1}{\pi^2 \hbar^3}}_{\text{Density of states}} \underbrace{\left(\frac{1}{3} v p\right)}_{\substack{\text{Radial pressure of particle with velocity } v \text{ and} \\ \text{momentum } p \text{ in a spherically symmetric system}}}$$

we have $p = \gamma m v$

$$= m \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m}{\sqrt{\frac{1}{v^2} - \frac{1}{c^2}}}$$

$$\Rightarrow \quad \frac{1}{v^2} - \frac{1}{c^2} = \frac{m^2}{p^2}$$

$$\Rightarrow \quad \frac{1}{v^2} = \frac{m^2}{p^2} + \frac{1}{c^2}$$

$$\Rightarrow v = \frac{pc}{\sqrt{m^2c^2+p^2}}$$

$$\Rightarrow P = \int_0^{p_F} dp\, p^2 \frac{1}{\pi^2 \hbar^3}\left(\frac{1}{3} p \frac{pc}{\sqrt{m^2c^2+p^2}}\right)$$

$$P = \frac{c}{3\pi^2\hbar^3}\int_0^{p_F} dp \frac{p^4}{\sqrt{m^2c^2+p^2}}$$

(i) In the ultra relativistic limit $p \gg mc$

$$\Rightarrow \boxed{P_{rel} = \frac{c}{3\pi^2\hbar^3}\int_0^{p_F} dp\, p^3 = \frac{c\, p_F^4}{12\pi^2\hbar^3}}$$

(ii) In the non relativistic limit

$$p << mc$$

$$\Rightarrow \boxed{P_{\text{non-rel}} = \frac{c}{3\pi^2\hbar^3}\int_0^{p_f} dp\, \frac{p^4}{mc} = \frac{p_f^{\,5}}{15\pi^2 m_e \hbar^3}}$$

using

$$n_e = \frac{1}{3}\frac{1}{\pi^2\hbar^3}\, p_f^{\,3}$$ from earlier

$$\Rightarrow \quad p_f = (3\pi^2\hbar^3)^{\frac{1}{3}}\, n_e^{\frac{1}{3}}$$

$$\Rightarrow \boxed{P_{\text{rel}} = \frac{c\,(3\pi^2\hbar^3)^{\frac{4}{3}}}{12\pi^2\hbar^3}\, n_e^{\frac{4}{3}}}$$

$$\Rightarrow \quad \boxed{P_{non\text{-}rel} = \frac{1}{15\pi^2 m_e \hbar^3}\left(3\pi^2\hbar^3\right)^{\frac{5}{3}} n_e^{\frac{5}{3}}}$$

Rewrite n_e in terms of mass density ρ.

Number of electrons per nucleon

$$n_e = \frac{Y_e \rho}{m_p} = \frac{\rho}{2m_p}$$

$= \frac{1}{2}$

$$\boxed{P_{rel} = \frac{c}{12\pi^2\hbar^3}\left(\frac{3\pi^2\hbar^3}{2m_p}\right)^{\frac{4}{3}} \rho^{\frac{4}{3}}}$$

$$\boxed{P_{non\text{-}rel} = \frac{1}{15\pi^2 m_e \hbar^3}\left(\frac{3\pi^2\hbar^3}{2m_p}\right)^{5/3} \rho^{5/3}}$$

But which is closer to the truth? Are the electrons in a white dwarf ultra relativistic, or non relativistic?

We will assume they are ultrarelativistic. (which at high densities, near the chandrasekhar limit turns out to be a good approximation)

White Dwarf Equation of State

Hence $$P = K\rho^{4/3}$$

where $$K = \frac{c}{12\pi^2\hbar^3}\left(\frac{3\pi^2\hbar^3}{2m_p}\right)^{4/3}$$

$$= 4900922906 = 4.90\times10^9$$

Recall two of the equations of stellar structure from 'The Sun & Stellar interiors' space walk.

$$\boxed{\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}}$$

Hydrostatic equillibrium equation

$$\boxed{\rho(r) = \frac{1}{4\pi r^2}\frac{dM}{dr}}$$

Mass continuity equation

Let's play with these to try and form a differential equation describing the density profile of the W.D.

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\frac{r^2}{\rho(r)}\frac{dP}{dr} = -GM(r)$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP}{dr}\right) = -G\frac{dM(r)}{dr}$$

replace using mass continuity equation

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP}{dr}\right) = -4\pi G r^2 \rho(r)$$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{1}{\rho(r)}\frac{dP}{dr}\right) = -4\pi G \rho(r)}$$

using EOS: $P = K\rho^{4/3}$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\,\frac{4}{3}K\,\frac{1}{\rho(r)}\,\rho(r)^{\frac{1}{3}}\,\frac{d\rho}{dr}\right) = -4\pi G\,\rho(r)$$

$$\boxed{\frac{1}{r^2}\,\frac{d}{dr}\left(r^2\frac{1}{\rho^{2/3}}\,\frac{d\rho}{dr}\right) = -\,\frac{3\pi G}{K}\,\rho(r)}$$

Differential equation for density profile

We will transform this equation into conveniently chosen dimensionless variables, ξ and Θ: (standard trick)

$$r = \xi\sqrt{\frac{K}{\pi G\rho_c^{2/3}}}\,, \quad \rho = \rho_c\left[\Theta(\xi)\right]^3$$

Core Pressure

$$\frac{1}{\xi^2}\cancel{\frac{\pi G \rho_c^{2/3}}{K}}\frac{d\xi}{dr}\frac{d}{d\xi}\left(\cancel{\frac{K}{\pi G \rho_c^{2/3}}}\xi^2\frac{3\cancel{\rho_c\Theta[\xi]^2}}{\rho_c^{2/3}\cancel{\Theta[\xi]^2}}\frac{d\xi}{dr}\frac{d\Theta}{d\xi}\right)$$

$$= -\frac{3\pi G}{K}\cancel{\rho_c}\,\Theta[\xi]^3$$

Note $$\frac{d\xi}{dr} = \sqrt{\frac{\pi G \rho_c^{2/3}}{K}}$$

$$\frac{1}{\xi^2}\left(\frac{\cancel{\pi G \rho_c^{2/3}}}{\cancel{K}}\right)\frac{d}{d\xi}\left(\xi^2\cancel{3}\,\cancel{\rho_c^{-\frac{2}{3}}}\frac{d\Theta}{d\xi}\right)$$

$$= -\frac{\cancel{3\pi G}}{\cancel{K}}\Theta^3$$

$$\Rightarrow \quad \boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^3}$$

Non-Dimensionalized Density Profile equation.

[Note: This is an example of the Lane-Emden equation, with $n=3$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n$$
]

We cannot solve this equation analytically for the W.D density profile, so will have to do it numerically.

first, what do we expect $\Theta(\xi)$ to look like?

$\xi \propto r$

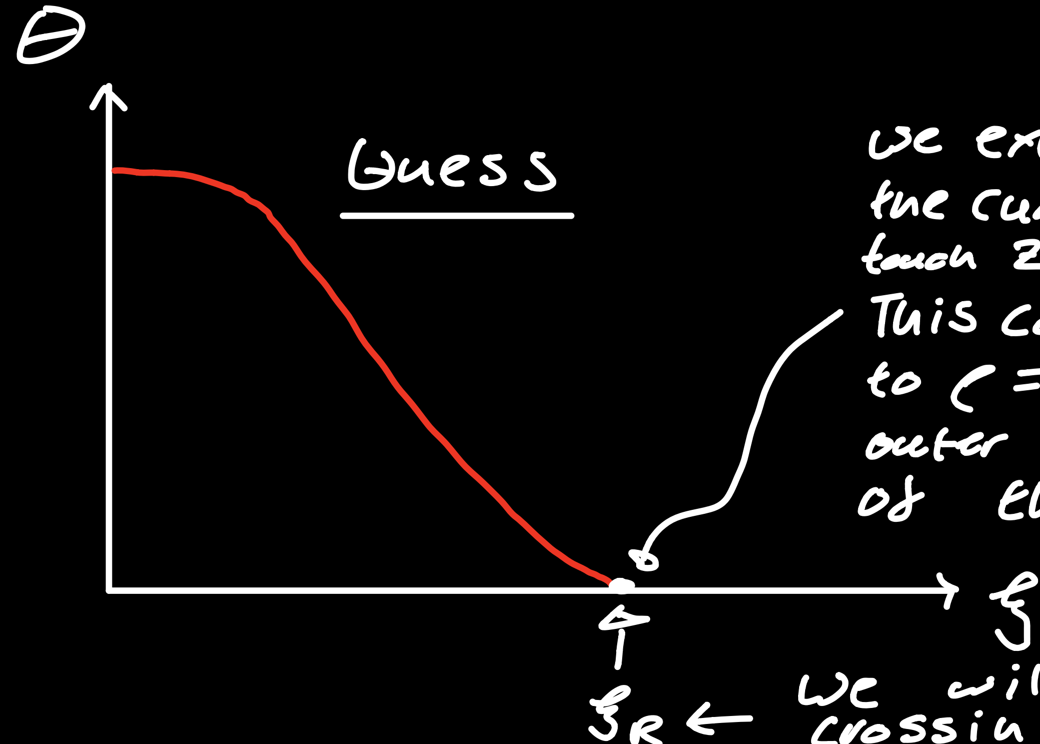
ξ measures radius

$$\Theta[\xi] = \left[\frac{\rho}{\rho_c}\right]^{\frac{1}{3}}$$

measures density compared to core density

We expect $\Theta(\xi = 0) = 1$ as $\rho = \rho_c$ at centre

We also expect $\frac{d\Theta}{d\xi}(\xi = 0) = 0$ for density to be non singular



We expect the curve to touch zero. This corresponds to $\rho = 0$ at the outer boundary of the star

$\xi_R \leftarrow$ We will call this crossing ξ_R

To Solve this, we will attempt the most naive numerical approach, which will turn out to be sufficient.

first let's change variable names, to make coding into MATLAB a little easier:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^3$$

$$y \equiv \theta$$

$$x \equiv \xi$$

And lets expand the Right hand side:

$$\boxed{\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^3 = 0}$$

with B.C'S: $y(0) = 1$

$$\frac{dy}{dx}(0) = 0$$

We now discretize the x axis for $x = 0$, to some upper limit (I will use $x = 10$) into equal timesteps of length Δx.

On this discretized x axis, J only has a value on the grid points shown.

We can label each grid point with an integer n denoting how many steps Δx we are away from zero.

We label the value of $y(x)$ at the n^{th} grid point as:

$$y_n := y(x_n) = y(n\Delta x)$$

In this scheme, we can rewrite our derivatives at the point x_n as:

$$\frac{dy}{dx}(x) = \frac{y_{n+1} - y_n}{\Delta x}$$

$$\frac{d^2y}{dx^2}(x) = \frac{y_{n+1} - 2y_n + y_{n-1}}{(\Delta x)^2}$$

This numerical scheme with fixed time step (Δx) and derivatives calculated in this way is called the explicit Euler method.

Let's substitute these discretized derivatives into our differential equation:

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^3 = 0$$

$$\Downarrow$$

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{(\Delta x)^2} + \frac{2}{x_n}\frac{y_{n+1} - y_n}{\Delta x} + y_n^3 = 0$$

$\leftarrow$ recall $x_n = n\Delta x$

Rearrange for y_{n+1}:

$$y_{n+1}\left[\frac{1}{(\Delta x)^2} + \frac{2}{n(\Delta x)^2}\right] = y_n\left[\frac{2}{n(\Delta x)^2} + \frac{2}{(\Delta x)^2}\right]$$

$$- y_{n-1}\left[\frac{1}{(\Delta x)^2}\right] - y_n^3$$

$$\Rightarrow y_{n+1} = \frac{\left[2 + \frac{2}{n}\right] y_n - y_n^3 - y_{n-1}}{\left[1 + \frac{2}{n}\right]}$$

This algebraic (No longer differential) equation gives us an approximation for the value of y at x_{n+1} given we know the values of y at x_n and x_{n-1}.

We must translate our boundary conditions $y(0) = 1$ and $\frac{dy}{dx}(0) = 0$ into values for y_0 and y_1.

Then using y_0 and y_1 we will find y_2, then using y_1 and y_2 we will find y_3 and so on....

$y(0) = 1 \Rightarrow \boxed{y_0 = 1}$

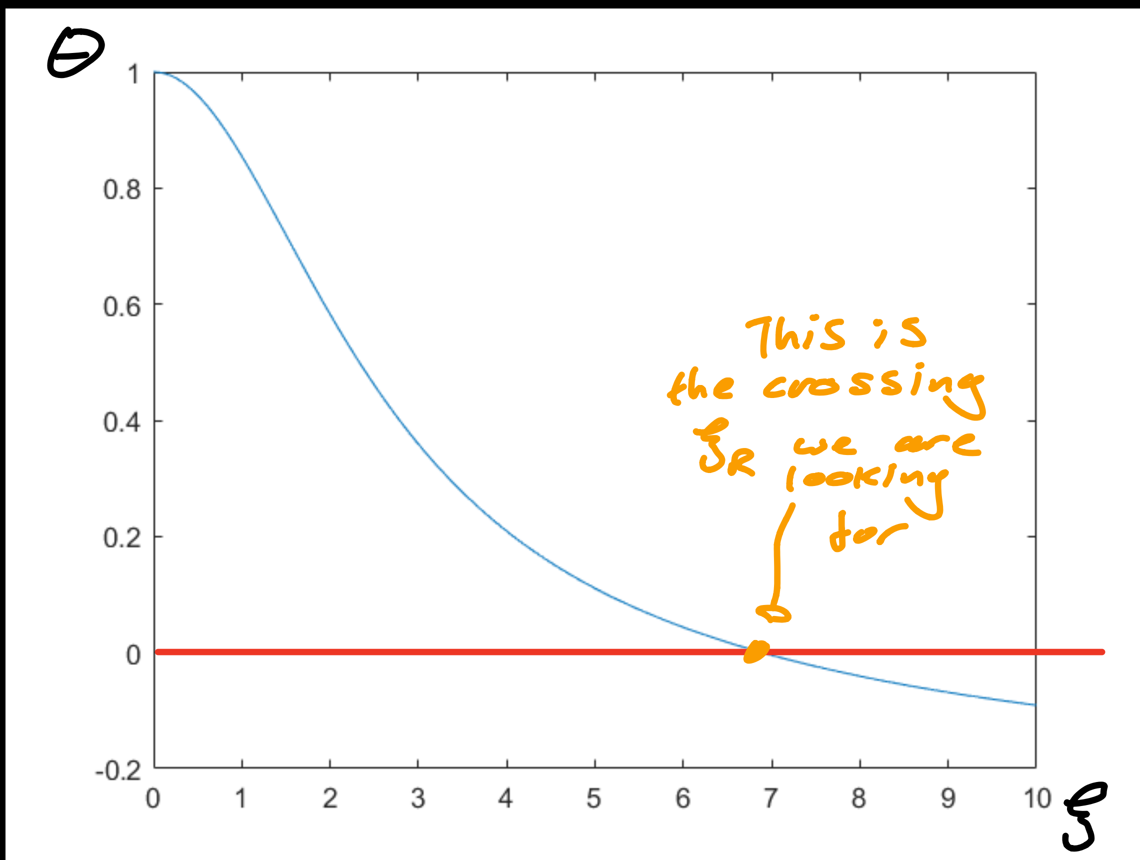
$\frac{dy}{dx}(0) = 0 \Rightarrow \frac{y_1 - y_0}{\Delta x} = 0 \Rightarrow \boxed{y_1 = y_0 = 1}$

We will write code to find all further y_n's.

```
%% Explicit Euler Method
clear all;
h = 0.002;
N = ceil(10./h);
y = zeros(1,N);

y(1) = 1;
y(2) = 1;

for n = 3:N;
    y(n)=((2+2./n).*y(n-1) - (h.^2).*y(n-1).^3 - y(n-2))./(1+2./n)
end

plot(h.*[1:N],y)
```

← note, I have renamed 'Δx' as 'h'

The plot of $y(x)$ [or $\Theta(\xi)$ to use the original names] looks like this:



Next, we find this crossing by interpolating:

```
%% Find Zero of Function

fmin = find(y < 0);
xmin = h.*(fmin(1));

fmax = find(y > 0);
xmax = h.*(fmax(length(fmax)));

xR = mean([xmin,xmax]);

%Result for h = 0.01: xR = 6.8450
%Result for h = 0.005: xR = 6.8725
%Result for h = 0.002: xR = 6.8870
```

The most accurate value for ξ_R I find is $\boxed{\xi_R = 6.887}$

Recall, we defined ξ as:

$$r = \xi \sqrt{\frac{K}{\pi G \rho_c^{2/3}}}$$

$\Rightarrow$ We have an expression for the radius of the white dwarf

$$R = \xi_R \sqrt{\frac{K}{\pi G \rho_c^{2/3}}}$$

[Remember, $K = 4.90 \times 10^9$]

$$\Rightarrow \boxed{R = 3.33 \times 10^{10} \rho_c^{-\frac{1}{3}}}$$

What we're searching for, is the maximum mass of a white dwarf.

From it's density profile (that we now know numerically), we can calculate its mass as follows:

$$M = \int_0^R dr\, 4\pi r^2 \rho(r)$$

Writing this in terms of our new variables ξ and $\theta(\xi)$:

$$r = \xi\sqrt{\frac{K}{\pi G \rho_c^{2/3}}}, \quad \rho = \rho_c\left[\theta(\xi)\right]^3$$

<u>or</u> $r = \frac{R}{\xi_R}\xi$

$$M = \int_0^{\xi_R} d\xi \, \frac{dr}{d\xi} \, 4\pi \xi^2 \frac{K}{\pi G \rho_c^{2/3}} \rho_c \left[\theta(\xi)\right]^3$$

$$= \left(\frac{K}{\pi G}\right)^{3/2} 4\pi \int_0^{\xi_R} \xi^2 \left[\theta(\xi)\right]^3 d\xi$$

$$= 1.42\times10^{30} \underbrace{\int_0^{\xi_R} \xi^2 \left[\theta(\xi)\right]^3 d\xi}_{\text{calculate this numerically}}$$

$$= 1.42\times10^{30} \, B$$

with $B = \int_0^{\xi_R} \xi^2 \left[\theta(\xi)\right]^3 d\xi$

```
%% Find Dimensionless Mass Integral

x = h.*[1:N];

B = cumtrapz(x,(x.^2).*y.^3);

B = B(length(fmax));

%Result for h = 0.002: B = 2.0136
```

$$\Rightarrow M = 1.42\times10^{30} \times 2.0136$$

$$= 2.859\times10^{30}\ \text{kg}$$

$$= \boxed{1.44\ M_\odot \quad 2.d.p}$$

This is the chandrasekhar mass. The maximum mass for a white dwarf star that can